Some new weighted Leavitt path algebra homomorphisms

Final presentation for mathematics honours

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2019

Outline

Some history

- The subject of Leavitt path algebras, or Lpas, is an active research area
- It is a new subject whose seminal papers came from two groups of authors:
  - Gene Abrams and Gonzalo Aranda Pino in *The Leavitt path algebra of a graph*
  - P. Ara, M. A. Moreno, and E. Pardo in *Nonstable K-theory for Graph Algebras*
- The work in those seminal papers was initiated around the same time in 2004
- However, these groups worked independently of one another
- The genesis of the subject is in earlier work during the 1950's by W. G. Leavitt in *The Module Type of a Ring*.

What I found difficult

- I did not have a strong background in algebra or analysis
- Lpas are defined using generators and relations
- It is not immediately clear that the presentation gives something non-zero
- In general it is difficult to find representations of Lpas in finite matrices
- It is not obvious, for example, how to know when any two given elements of an Lpa are equal
- There is, however, a representation for any Lpa inside infinite dimensional matrices over a field

What went well?

- I wrote some software in the Haskell language to help me try to understand Lpas
- Actually I worked with weighted Lpas, or wLpas, which are a generalisation of Lpas due to Roozbeh Hazrat.
- My software can do the following things
  - Can check if two elements of an wLpa are equal
  - Caveat: I did not yet prove termination of my algorithm for checking equality!
  - Can enumerate the basis elements of a wLpa
  - Can check if a homomorphism defined on the generators of a wLpa is well-defined
  - Can compute some wLpa invariants
- I used this software to help me obtain some new homomorphisms

The mathematics

- What are Leavitt path algebras?
- I will give a formal definition shortly
- The subject combines graph theory and ring theory
- The finite dimensional case has a more combinatorial flavour
- The general case seems to be more about ring theory

The basics

- Let E be a directed graph
- Denote the vertices of E by E^0
- Denote the edges of E by E^1
- For any edge e define the *source* $s(e)$ as the vertex at the start of the edge
- For any edge e define the *range* $r(e)$ as the vertex at the end of the edge
- For any edge e define the "ghost-edge" e^*

The basics continued

- A *sink* is a vertex with no outgoing edges
- A *regular* vertex is a vertex that is not a sink
- Denote the set of regular vertices in a directed graph E by $\mathrm{Reg}(E)$
- Consider the oriented n-line graph

$$v_1 \xrightarrow{e_1} v_2 \xrightarrow{e_2} v_3 \cdots\cdots\cdots\cdots v_{n-1} \xrightarrow{e_{n-1}} v_n$$

- Then v_n is a sink and $v_1, \cdots, v_{n-1}$ are regular.

The formal definition

- The Leavitt path algebra on a graph is the free K-algebra generated by the vertices, edges, and ghost-edges, modulo the following relations

  V $uv = \delta_{uv} v$ for all $u, v \in E^0$

  E1 $s(e)e = er(e) = e$ for all $e \in E^1$

  E2 $r(e)e^* = e^* s(e) = e^*$ for all $e^* \in (E^1)^*$

  CK1 $e^* f = \delta_{ef} r(e)$ for all $e, f \in E^1$

  CK2 $v = \sum_{\substack{e \in E^1 \\ s(e)=v}} ee^*$ for all $v \in \mathrm{Reg}(E)$

- What is a free K-algebra?
- It is like a non-commutative version of a polynomial ring with coefficients from a field K

The formal definition continued

- What does modulo mean?
- It means quotient
- What is a quotient?
  - A quotient is a way to obtain a new algebra from an old one by including some relations
  - It is defined in most algebra texts using set theory.

The motivation

- Why these relations and what do they mean?
- In the case of finite directed acyclic graphs these relations describe an algebra with some interesting combinatorial properties
- What does acyclic mean?
- It means there are no cycles
- What is a cycle?
- A *cycle* in a directed graph is a path $e_1, \cdots, e_n$ such that $s(e_i) \neq s(e_j)$ for every $i \neq j$, and $r(e_n) = s(e_1)$

The motivation continued

- What is a path?
- A *path* in a directed graph is a finite sequence of edges $e_1, \cdots, e_n$, such that $r(e_i) = s(e_{i+1})$ for all $1 \leq i \leq n-1$.
- Note: If we are discussing the Lpa of a graph, then the words path and cycle refer to the corresponding products of these edges, rather than merely sequences of edges

The oriented n-line graph

- Let us revisit the oriented n-line graph example,

$$v_1 \xrightarrow{e_1} v_2 \xrightarrow{e_2} v_3 \cdots\cdots\cdots v_{n-1} \xrightarrow{e_{n-1}} v_n$$

- We want to be able to add, multiply, and subtract vertices, edges, and ghost-edges
- How do we define this in a meaningful way?
- Use matrices
- We represent the vertex v_1 as the matrix

$$\begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

The oriented n-line graph continued

- We represent the vertex v_2 as the matrix

$$\begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \hdotsfor{5} \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

- See the pattern?
- Represent vertex v_k as the single-entry matrix with 1 in the kth diagonal entry
- What about edges?

The oriented n-line graph continued

Representations for the edges are as follows,

- We represent the edge e_1 as the matrix

$$\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \hdotsfor{5} \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

- We represent the edge e_2 as the matrix

$$\begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \hdotsfor{5} \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

The oriented n-line graph continued

- We represent the edge e_k as the single-entry matrix with 1 in the kth row and $(k+1)$th column
- What about ghost-edges?
- The $*$ operation corresponds to the matrix transpose or adjoint
- For example e_1^* is represented by

$$\begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ \hdotsfor{5} \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

- The ghost-edge e_k^* is represented by the single-entry matrix with 1 in the $(k+1)$th row and the kth column

The oriented n-line graph continued

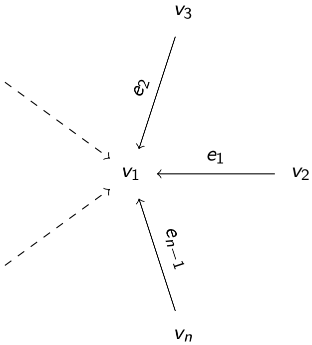
- Let $M_K(n)$ be the K-algebra of square n by n dimensional matrices over the field K
- For any directed graph E we will denote the Leavitt path algebra of E with coefficients from the field K by $L_K(E)$
- For the oriented n-line graph, A_n, we get $L_K(A_n) \cong M_K(n)$
- What about other directed acyclic graphs?

Directed acyclic graphs

- Consider any directed acyclic graph E with a sink v
- Since the graph is finite acyclic, there must be a finite number of paths that end at v. Let this number be $n(v)$.
- The ideal generated by v, $I(v)$ is isomorphic to $M_K(n(v))$
- If there is only one sink, then this ideal is the entire Lpa
- If there is more than one sink, then the Lpa is given by the direct product of the ideals generated by the sinks.

Directed acyclic graphs continued

- Consider this example,



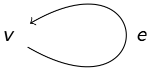
- Denote the above graph with n vertices as B_n
- Denote the oriented n-line graph by A_n
- Then $L_K(B_n) \cong L_K(A_n) \cong M_K(n)$

Directed acyclic graphs continued

- For any $n > 0$ there are a finite number of directed acyclic graphs whose Lpa is isomorphic to $M_K(n)$
- Counting these graphs appears to be a non-trivial problem
- This problem seems to be worthy of further study
- I was not able to find anything in the literature about this so far

Graphs with cycles

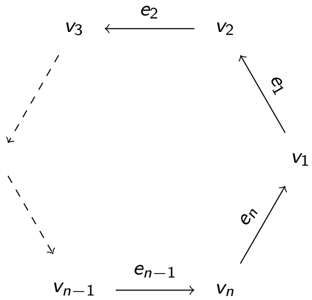
- For certain types of graphs with cycles, we may still be able to obtain representations in finite matrices over Laurent polynomial rings
- Consider the loop graph



- The Lpa for this graph is isomorphic to $K[x, x^{-1}]$, the Laurent polynomial ring over K

Graphs with cycles continued

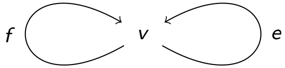
- Or the n-cycle graph



- The Lpa for this graph is isomorphic to $M_{K[x,x^{-1}]}(n)$

Graphs with cycles continued

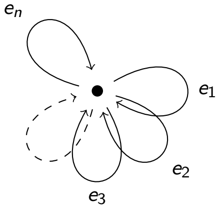
- Consider the 2-petal graph



- The Lpa of this graph, call it R, has an unusual property
- As left R-modules, we have $R \cong R \oplus R$
- A *module* is like a vector space over a ring
- W. G. Leavitt was the first person to produce examples of rings having the above property

Graphs with cycles continued

- More generally, consider the n-petal graph for some fixed integer $n > 0$,



- Denote by R^n the direct sum $\underbrace{R \oplus \cdots \oplus R}_{n \text{ times}}$
- The Lpa of this graph, call it R, satisfies $R \cong R^n$
- Moreover this n is the smallest integer m such that $R \cong R^m$

Equality

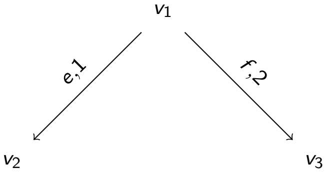
- In the general case of an Lpa, how do we know if two given elements of the algebra are equal?
- This is a non-trivial issue since we may not necessarily have a finite representation for the algebra
- There is a theorem that gives a *basis* for any Lpa, viewed as a K-module
- Checking equality amounts to reducing elements to their basis form and comparing them
- This basis will only be finite if the graph is finite acyclic

A Haskell implementation

- I coded an algorithm using the Haskell language to check for equality of wLpa elements
- The algorithm works as follows,
  1. First convert the given element of the wLpa to normal form. That is, a K-linear combination of paths
  2. For each path in this expression, find any *forbidden* sub-paths. The term forbidden is a technical term!
  3. For each forbidden sub-path found, rewrite this sub-path using the wLpa relations to remove it
  4. Go back to the start and iterate this procedure until no forbidden sub-paths remain. Then stop.
- A path that contains no forbidden sub-paths is called a *nod-path*
- Proving termination of the above algorithm would show that nod-paths are a spanning set for the wLpa

Weighted graphs

- We assign positive integer weights to the edges of a directed graph. For example (E, w) given as,



- Here, E is the unweighted directed graph and w is a function that maps $E^{st} \to \mathbb{Z}^+$
- So in this example $w(e) = 1$ and $w(f) = 2$
- The vertex set is given by

$$E^0 = \{v_1, v_2, v_3\}$$

- The edge set is given by

$$E^1 = \{e_1, f_1, f_2\}$$

Weighted graphs continued

The formal definition is as follows,

- Let E^0 be a countable set called *vertices*
- Let E^{st} be a countable set called *structured edges*
- Let there be maps $s, r : E^{st} \to E^0$
- Let there be a *weight map* $w : E^{st} \to \mathbb{Z}^+$
- Define the *edges* as the set $E^1 = \cup_{\alpha \in E^{st}} \{\alpha_i \mid 1 \leq i \leq w(\alpha)\}$

Weighted Leavitt path algebras

The formal definition,

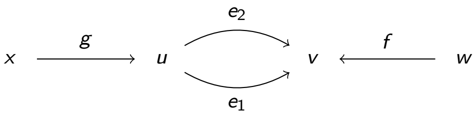
- A wLpa is the free K-algebra on $E^0 \cup E^1 \cup (E^1)^*$, subject to the following relations
  1. $v_i v_j = \delta_{ij} v_i$ for every $v_i, v_j \in E^0$.
  2. $s(\alpha)\alpha_i = \alpha_i r(\alpha) = \alpha_i$ and $r(\alpha)\alpha_i^* = \alpha_i^* s(\alpha) = \alpha_i^*$ for all $\alpha \in E^{st}$ and $1 \leq i \leq w(\alpha)$.
  3. $\sum_{\{\alpha \in E^{st} \mid s(\alpha)=v\}} \alpha_i \alpha_j^* = \delta_{ij} v$ for fixed $1 \leq i, j \leq \max\{w(\alpha) \mid \alpha \in E^{st}, s(\alpha) = v\}$, for all $v \in E^0$.
  4. $\sum_{1 \leq i \leq \max\{w(\alpha), w(\alpha')\}} \alpha_i^* \alpha_i' = \delta_{\alpha\alpha'} r(\alpha)$, for all $\alpha, \alpha' \in E^{st}$.
- In the above, if $e \in E^{st}$ is a structured edge and $i > w(e)$ then $e_i = 0$.

Some wLpa examples

- Many simple examples of wLpas are isomorphic to unweighted Lpas
- For example the wLpa of

$$x \xrightarrow{g,1} u \xleftarrow{e,2} v \xleftarrow{f,1} w$$

- is isomorphic to the Lpa of



Some wLpa examples continued

The isomorphism is given on the generators as,

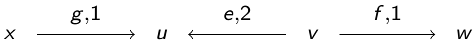
$$\begin{aligned} u &\mapsto u & \quad e_1 &\mapsto e_1^* \\ v &\mapsto v & \quad e_2 &\mapsto e_2^* \\ w &\mapsto w & \quad f_1 &\mapsto f \\ x &\mapsto x & \quad g_1 &\mapsto g \end{aligned}$$

Some wLpa examples continued

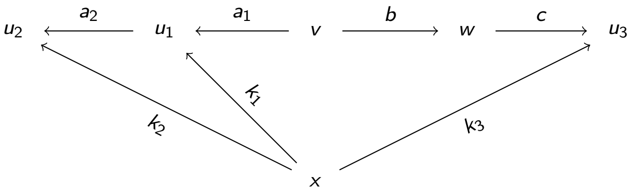
- To see this is a well defined homomorphism,
  1. Start with the free algebra on the vertices, edges, and ghost edges of the weighted graph
  2. As soon as you have a map from the generators to a ring, it induces a homomorphism on the level of algebra, by the properties of a free algebra
  3. Next one needs to check that the weighted relations will be mapped to zero in the Lpa.
  4. This guarantees a map from the quotient of the free algebra modulo the weighted relations into the Lpa
- Step 3 can be checked by computer.

Some wLpa examples continued

- The wLpa of



- is isomorphic to the Lpa of



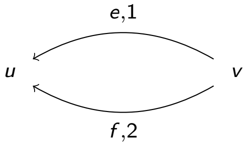
Some wLpa examples continued

The isomorphism is given on the generators as,

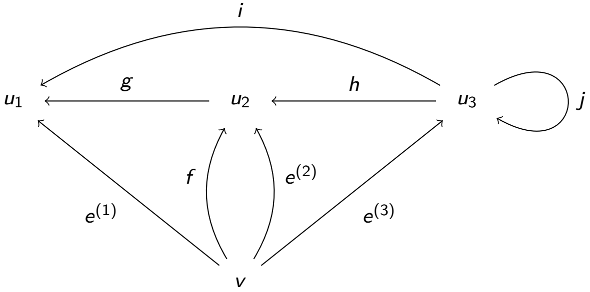
$$
\begin{aligned}
u &\mapsto u_1 + u_2 + u_3 & e_1 &\mapsto a_1 \\
v &\mapsto v & e_2 &\mapsto a_1 a_2 + bc \\
w &\mapsto w & f_1 &\mapsto b \\
x &\mapsto x & g_1 &\mapsto k_1 + k_2 + k_3
\end{aligned}
$$

Some wLpa examples continued

- A more complicated example. The wLpa of



- this is isomorphic to the Lpa of



Some wLpa examples continued

The isomorphism is given on the generators as,

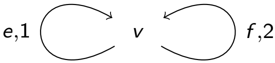
$$u \mapsto u_1 + u_2 + u_3 \qquad e_1 \mapsto e_1 + e_2 + e_3$$

$$v \mapsto v \qquad f_1 \mapsto f$$

$$f_2 \mapsto fg + e_1 i^* + e_2 h^* + e_3 j^*$$

An unsolved problem

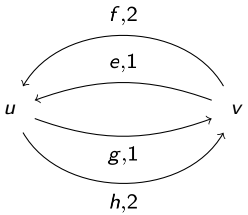
- The previous example is from a paper in arXiv entitled *The V-monoid of a weighted Leavitt path algebra* by Raimund Preusser
- In the same paper Preusser gives the following example



- At the present time it is unknown whether this wLpa is isomorphic to an unweighted Lpa

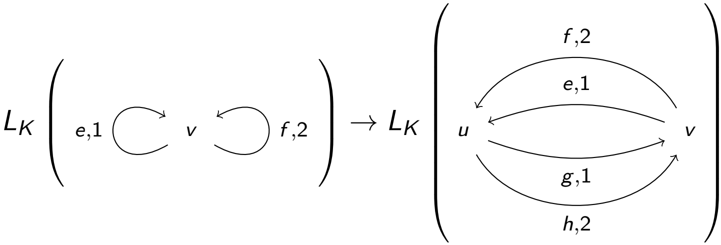
Some new homomorphisms

- In my honours thesis I introduced the following weighted graph,



- I was able to obtain some new homomorphisms using this

Some new homomorphisms continued

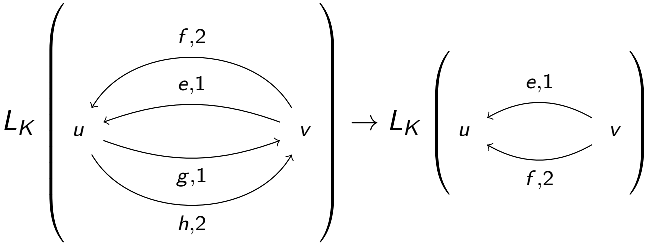


$$v \mapsto v + u \qquad f_1 \mapsto f_1 + h_1$$

$$e_1 \mapsto e_1 + g_1 \qquad f_2 \mapsto f_2 + h_2$$

- Call this map η
- I claim that η is a monomorphism.

Some new homomorphisms continued



$$
\begin{array}{llll}
u \mapsto u & e_1 \mapsto e_1 & f_2 \mapsto f_2 & h_1 \mapsto f_1^* \\
v \mapsto v & f_1 \mapsto f_1 & g_1 \mapsto f_2^* & h_2 \mapsto e_1^*
\end{array}
$$

- Call this map ρ
- Observe that the generators of the codomain all appear in the range of ρ
- Therefore ρ is an epimorphism

Some new homomorphisms continued

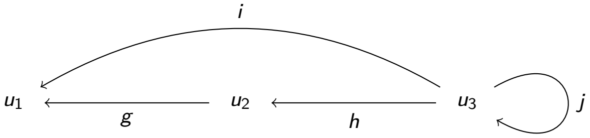
- Define $Q = \{h_2^* - e_1, h_1^* - f_1, g_1^* - f_2\}$
- In my honours thesis I show that $\ker(\rho) = I(Q)$
- Let $\theta = \rho \circ \eta$
- So θ is given by

$$\begin{aligned} v &\mapsto v + u & \quad f_1 &\mapsto f_1 + f_1^* \\ e_1 &\mapsto e_1 + f_2^* & \quad f_2 &\mapsto f_2 + e_1^* \end{aligned}$$

- Let $R = \{e_1 - f_2^*, f_1 - f_1^*\}$
- I claim that $\ker(\theta) = I(R)$

Some new homomorphisms continued

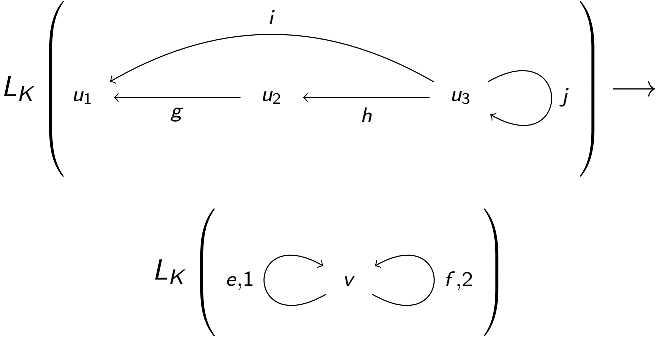
- In my honours thesis I introduced the following unweighted graph



- I was able to obtain some new homomorphisms using this

Some new homomorphisms continued

There is a homomorphism,



Some new homomorphisms continued

- It is given on the generators by

$$u_1 \mapsto x^* x \qquad g \mapsto x$$
$$u_2 \mapsto x x^* \qquad h \mapsto y - yz$$
$$u_3 \mapsto y^* y - x^* x - x x^* \qquad i \mapsto y(z - yy^*)$$
$$j \mapsto y^2 y^*$$

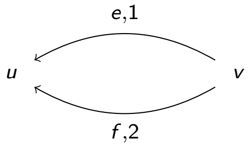
where,

$$x = f_1^* f_2 \qquad y = f_2^* e_1 \qquad z = f_2^* f_2$$

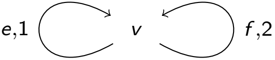
- I claim this map is mono

Some new homomorphisms continued

There is a representation in 2 by 2 matrices as follows,

L_K () $\hookrightarrow M_R(2)$

where

$R = L_K$ ()

Some new homomorphisms continued

It is given on the generators as,

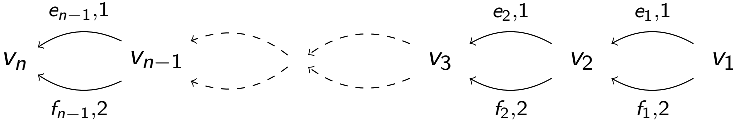
$$e_1 \mapsto \begin{bmatrix} 0 & e_1 \\ 0 & 0 \end{bmatrix} \qquad u \mapsto \begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}$$

$$f_1 \mapsto \begin{bmatrix} 0 & f_1 \\ 0 & 0 \end{bmatrix} \qquad v \mapsto \begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}$$

$$f_2 \mapsto \begin{bmatrix} 0 & f_2 \\ 0 & 0 \end{bmatrix}$$

Some new homomorphisms continued

- More generally, let (G_n, w_{G_n}) be the following weighted graph,



- There is a representation,

$$L_K(G_n, w_{G_n}) \hookrightarrow M_R(n)$$

for all $n > 1$

Some new homomorphisms continued

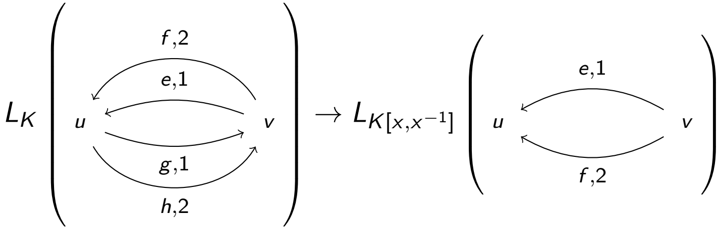
- It is given by,

$$e_{i,1} \mapsto e_1 c_{i,i+1} \qquad f_{i,1} \mapsto f_1 c_{i,i+1} \qquad f_{i,2} \mapsto f_2 c_{i,i+1}$$
$$v_i \mapsto v c_{i,i}$$

  where $c_{i,j}$ is the n by n single-entry matrix with 1 in the ith row and jth column.
- Question: how to generalise this result further?

Some new homomorphisms continued

- There is a homomorphism as follows,



- Note that in the codomain the coefficients come from a Laurent polynomial $*$-ring
- We assume that $x^* = x^{-1}$

Some new homomorphisms continued

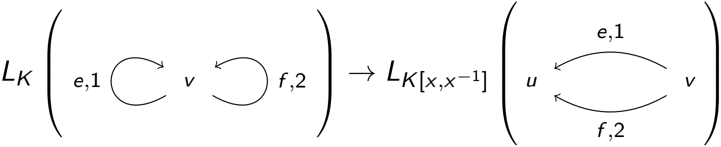
- It is given on the generators as,

$$\begin{aligned} e_1 &\mapsto xe_1 & g_1 &\mapsto xf_2^* & & \\ f_1 &\mapsto xf_1 & h_1 &\mapsto xf_1^* & u &\mapsto u \\ f_2 &\mapsto xf_2 & h_2 &\mapsto xe_1^* & v &\mapsto v \end{aligned}$$

- I claim this map is mono

Some new homomorphisms continued

- Composing an earlier example with the above example gives a monomorphism,

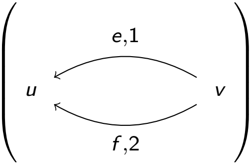


- The map is given by,

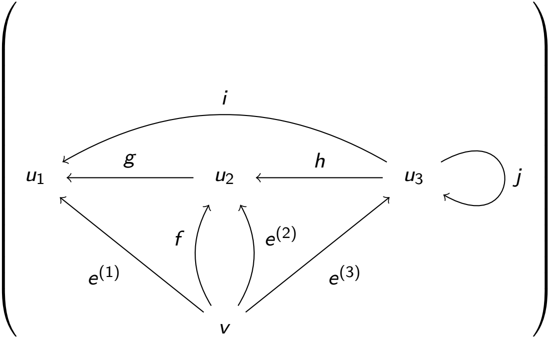
$$e_1 \mapsto x(e_1 + f_2^*)$$
$$f_1 \mapsto x(f_1 + f_1^*) \qquad v \mapsto u + v$$
$$f_2 \mapsto x(f_2 + e_1^*)$$

Some new homomorphisms continued

Recall that,

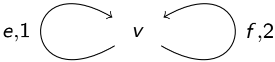
L_K 

$\xrightarrow{\sim}$

L_K 

Some new homomorphisms continued

- It follows that our wLpa,

$L_K\Big($  $\Big)$

- is sitting inside an unweighted Lpa over a Laurent polynomial $*$-ring

Where is the code?

- The Haskell code I used is freely available on Github at

  `https://github.com/rzil/honours/tree/master/LeavittPathAlgebras`
- All of the above homomorphisms can be verified computationally using this code
- Coming up with these maps, I found enjoyable. I would encourage you to try it!

Further work

- In the above mentioned examples, there were a number of claims yet to receive proof
- For example, showing that certain maps are mono or finding generating sets for the kernels
- One idea for showing some of the injectivity claims, suggested by R. Preusser, is to show that nod-paths are mapped to distinct nod-paths
- Most of the maps given above were verified using computer. I would like to prove some of them on paper.
- The basis algorithm described earlier has not been proven to terminate. I would like to prove this.

Thank you for listening!