Leavitt Path Algebras

Some new homomorphisms and representations obtained using computational techniques

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Dedication

To my grandfather, Maximilian Henner, who is ninety-nine years old this year.

Contents

Chapter 1

Introduction

Throughout the course of this document various terminology is defined as appropriate. Where terminology is not explicitly defined the reader is referred to Appendix B where the necessary background material is defined. For any terminology not included in Appendix B the reader is referred to any standard book on algebra such as [8].

Most the the work contained here has been adapted from the academic literature for my purposes. I derived some of the proofs myself and therefore they might differ from the literature slightly. There are some original results that make some progress with a currently unsolved problem; see section 5.2

The subject of Leavitt path algebras is an active research area in the theory of rings, modules, and algebras. It is a new subject whose seminal papers came from two groups of authors, Gene Abrams and Gonzalo Aranda Pino in [2] and P. Ara, M. A. Moreno, and E. Pardo in [3]. The work in those seminal papers was initiated around the same time in 2004, however, these groups worked independently of one another. The genesis of the subject is in earlier work during the 1950's by W. G. Leavitt in [10]. In this thesis I attempt to provide an overview of the subject of Leavitt path algebras. I survey some of the latest work being done as well as some of the older results in the field. Due to my background in computing I made use of software and wrote software during my investigations too. I found this to be helpful as carrying out some of the necessary computations by hand would be difficult and error prone. I would also speculate that the use of this kind of software could help in teaching the subject.

For any module, ring, or field, R denote $\underbrace{R \oplus \cdots \oplus R}_{n \text{ times}}$ as R^n. If we have a field $\mathbb{F}$ and two positive integers n and m such that $\mathbb{F}^n \cong \mathbb{F}^m$ as $\mathbb{F}$-modules then we may conclude that $n = m$. Note that an $\mathbb{F}$-module is a vector space. This property is known as the Invariant Basis Number or IBN. It is true of vector spaces and until the 1950's it was not known if general modules always obeyed this property or not. Leavitt was the first person to produce examples of non-IBN K-algebras R for a field K, where, when viewed as left R-modules $R^n \cong R$ for a fixed $n > 1$. Leavitt path algebras or Lpas, which are the topic of this thesis, are a generalisation of Leavitt's original construction. Lpas can be associated to directed graphs.

Other prominent figures in the history of this subject are Bergman and Cohn. Bergman's work helped to understand the graph monoid which we define and study in chapter 4. The graph monoid gives us information about the whether the algebra has the IBN property. We also introduce Cohn path algebras in chapter 4 as these provide a way to construct Lpa examples having IBN.

There are various ways to generalise Lpas. One such generalisation was discovered by Roozbeh Hazrat. This is the weighted Leavitt path algebra or wLpa. These objects capture even more examples of non-IBN algebras. I developed software in the Haskell programming language to carry out computations relating to wLpas. This and various results about wLpas are explored in chapter 5. A link to the github repository for the source code is also included in chapter 5.

As a simple motivating example, consider the ring of column finite matrices, with countably many rows and columns, over a ring R. We have the following module isomorphism

$$\begin{aligned} \psi : \mathbb{CFM}(R) &\to \mathbb{CFM}(R)^2 \\ M &\mapsto (\text{odd columns of } M, \\ &\qquad \text{even columns of } M) \end{aligned} \tag{1.1}$$

It is not too hard to verify the above claim by inspection. More formally, define

$$A = \begin{bmatrix} 1 & 0 & 0 & 0\dots \\ 0 & 0 & 0 & 0\dots \\ 0 & 1 & 0 & 0\dots \\ 0 & 0 & 0 & 0\dots \\ 0 & 0 & 1 & 0\dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Then equation (1.1) reads

$$\begin{aligned} \psi : \mathbb{CFM}(R) &\to \mathbb{CFM}(R)^2 \\ M &\mapsto (MA, MB) \end{aligned} \tag{1.2}$$

We immediately see that ψ is a homomorphism due to the linearity of matrix multiplication. Also we have

$$\begin{aligned} \psi^{-1} : \mathbb{CFM}(R)^2 &\to \mathbb{CFM}(R) \\ (X, Y) &\mapsto XA^T + YB^T \end{aligned} \tag{1.3}$$

as one can verify by direct computation. Since ψ^{-1} is also bilinear it is a homomorphism. This shows that ψ is indeed an isomorphism.

Definition 1.0.1. Let the *natural numbers*, denoted by $\mathbb{N}$, be the commutative semiring consisting of the non-negative integers. That is, $\mathbb{N} = \{0, 1, \cdots\}$.

Definition 1.0.2. Suppose we have a ring R which is not IBN. Let $m \in \mathbb{N}$ be minimal with the property that $R^m \cong R^{m'}$ as left R-modules for some $m' > m$. For this m, let n denote the minimal such m'. In this case we say that R has *module type* (m, n).

In [10] W. G. Leavitt constructs explicit examples of these, proving,

Theorem 1.0.1. *For each pair of positive integers $n > m$ and field K there exists a unital K-algebra $L_K(m, n)$ that has module type (m, n).*

We call the algebras constructed by W. G. Leavitt proving the above theorem, Leavitt algebras of type (m, n). The four matrices given above, namely $\{A, A^T, B, B^T\}$, with coefficients from an arbitrary field K, generate precisely the Leavitt algebra of type $(1, 2)$. More on this is said in chapter 5.

Chapter 2

The Finite Dimensional Case

We start by looking at the simplest case of an Lpa, the finite dimensional case. This case already involves a considerable amount of mathematics. It will help set the stage for studying Lpas more generally.

We begin with some notation and terminology for describing the structure of a directed graph. This is standard notation in Lpa literature.

Definition 2.0.1. Let E be any directed graph. Throughout this document we will refer to the set of vertices in E as E^0, and the set of edges in E as E^1. Also for any edge e going from vertex v to vertex w, we will call v the source of e denoted $s(e)$, and w the range of e denoted $r(e)$. A vertex is called a *source* if it has no incoming edges. A vertex is called a *sink* if it has no outgoing edges. A non-sink vertex is called *regular*. We say that E is *row-finite* if the set of outgoing edges for every vertex in E is finite. We say that E is *finite* if the set of edges if finite and the set of vertices is also finite. The set of regular vertices in E is denoted as $\text{Reg}(E)$.

Paths are defined as follows

Definition 2.0.2. Let E be any directed graph. Let $e_1 \dots e_k$ be any list of edges in E^1 so that $r(e_i) = s(e_{i+1})$ for all $1 \leq i < k$. Then $\mu = e_1 \dots e_k$ is said to be a *path* in E. Define $\mu_i = e_i$, $s(\mu) = s(e_1)$, and $r(\mu) = r(e_k)$.

Cycles play an important role in the study of Lpas

Definition 2.0.3. A path μ is called a *cycle* if $s(\mu) = r(\mu)$ and $s(\mu_i) \neq s(\mu_j)$ for every $i \neq j$. A graph without cycles is said to be *acyclic*.

Now we give the definition of a Leavitt path algebra with its five relations.

Definition 2.0.4. Let E be any directed graph and K any field. Define $(E^1)^* = \{e^* \mid e \in E^1\}$. Let X be the free K-algebra generated by the set $E^0 \cup E^1 \cup (E^1)^*$. The Leavitt path algebra of E with coefficients in K, denoted $L_K(E)$ is the quotient of X by the following relations

V $uv = \delta_{uv}v$ for all $u, v \in E^0$

E1 $s(e)e = er(e) = e$ for all $e \in E^1$

E2 $r(e)e^* = e^*s(e) = e^*$ for all $e^* \in (E^1)^*$

CK1 $e^*f = \delta_{ef}r(e)$ for all $e, f \in E^1$

CK2 $v = \sum_{\substack{e \in E^1 \\ s(e)=v}} ee^*$ for all $v \in \mathrm{Reg}(E)$

We can prove a simple theorem about the identity of a finite Lpa.

Theorem 2.0.1. *Let E be a graph with finite vertices. Then $L_K(E)$ has an identity i and $i = \sum_{v \in E^0} v$.*

Proof. Any element of $L_K(E)$ can be written as a sum of monomials of the form $kpq*$ where $k \in K$ and p and q are paths in E. This follows from relation CK1. Let $j = \sum_{v \in E^0} v$. It follows from E1 that $jkpq^* = kpq^*$. It follows from E2 that $kpq^*j = kpq^*$. It immediately follows that j must be the identity. □

For the subsequent lemmas we need to use the fact that vertices of an Lpa are linearly independent. We state this theorem here without proof.

Theorem 2.0.2. *Let E be a directed graph. Let K be a field. Let X be the free K-algebra on the set $E^0 \cup E^1 \cup (E^1)^*$. Let $\pi : X \to L_K(E)$ be the natural quotient map. That is, $\pi(g) = g$ for every $g \in E^0 \cup E^1 \cup (E^1)^*$. Then π sends the vertices from E^0 to K-linearly independent elements of $L_K(E)$.*

Proof. This can be shown by constructing a representation of the Lpa in the endomorphism ring of a vector space. For details of this proof see Lemma 1.5 in [4]. □

We may now proceed to prove some facts about the Lpas of certain types of graphs.

Lemma 2.0.1. *Let E be a finite acyclic graph with a sink v. Then the monomials of the form $pq^* \in L_K(E)$, where $r(p) = r(q) = v$ are linearly independent.*

Proof. Since E is finite acyclic there must be a finite number of distinct paths p with $r(p) = v$. Let these paths be $p_1 \dots p_n$ where n is a positive integer. Observe that for any such paths p_i and p_j, we have $p_i^* p_j = \delta_{ij} v$.

Suppose we have

$$\sum_{i=1}^{n} \sum_{j=1}^{n} k_{ij} p_i p_j^* = 0$$

where the $k_{ij} \in K$ are fixed and not all the k_{ij} are zero. Then for any $1 \le a, b \le n$

$$p_a^* \left(\sum_{i=1}^{n} \sum_{j=1}^{n} k_{ij} p_i p_j^* \right) p_b = 0$$

However

$$p_a^* \left(\sum_{i=1}^{n} \sum_{j=1}^{n} k_{ij} p_i p_j^* \right) p_b = k_{ab} v$$

Therefore $k_{ab} v = 0$ for all $1 \le a, b \le n$. Since not all k_{ij} are zero by assumption this means $v = 0$. By theorem 2.0.2 vertices cannot be zero so this is a contradiction. Therefore the assumption that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} k_{ij} p_i p_j^* = 0$$

is false. □

We study certain ideals

Lemma 2.0.2. *Let E be a finite acyclic graph with a sink v. Let n be the number of paths ending at v. Then given any field K, the ideal of $L_K(E)$ generated by v satisfies $I(v) \cong M_n(K)$, where $M_n(K)$ is the K-algebra of n by n matrices over the field K.*

Proof. Note first that n is well defined since the graph is finite acyclic. Let the paths ending at v be $p_1 \dots p_n$. Observe that any monomial in $I(v)$ must be of the form $k p_i p_j^*$ for some $1 \le i, j \le n$, where $k \in K$.

Consider the element $s = \sum_{i=1}^{n} p_i p_i^*$. Observe that for any $1 \leq i, j, r \leq n$ we have that $(p_i p_j^*)(p_r p_r^*) = \delta_{jr} p_i p_r^*$ and also $(p_r p_r^*)(p_i p_j^*) = \delta_{ri} p_r p_j^*$. It follows that s is the multiplicative identity in $I(v)$. Therefore $I(v)$ is a K-algebra.

From lemma 2.0.1 the paths of the form $p_i p_j^*$ are linearly independent in $L_K(E)$. Hence they must also be linearly independent in $I(v)$. It is clear they also span $I(v)$. Therefore they form a finite K-basis for $I(v)$. Define

$$\begin{aligned} \phi : I(v) &\to M_n(K) \\ p_i p_j^* &\mapsto e_{i,j} \end{aligned}$$

Where $e_{i,j}$ is the standard matrix unit in row i column j. Since ϕ is defined on the basis we can see it is a homomorphism by observing that

$$\begin{aligned} \phi((p_i p_j^*)(p_a p_b^*)) &= \phi(\delta_{j,a} p_i p_b^*) \\ &= \delta_{j,a} \phi(p_i p_b^*) \\ &= \delta_{j,a} e_{i,b} \\ &= e_{i,j} e_{a,b} \\ &= \phi(p_i p_j^*) \phi(p_a p_b^*) \end{aligned}$$

We also define

$$\begin{aligned} \psi : M_n(K) &\to I(v) \\ e_{i,j} &\mapsto p_i p_j^* \end{aligned}$$

It is easy to see that ψ is a homomorphism in a similar way. It is also clear that $\phi \circ \psi$ and $\psi \circ \phi$ are the identity maps on the respective domains. It follows that ϕ is an isomorphism. Therefore $I(v) \cong M_n(K)$. □

Example 2.0.1. For any positive integer n we let A_n denote the *oriented n-line graph* having n vertices and $n-1$ edges:

$$A_n = \quad v_1 \xrightarrow{e_1} v_2 \xrightarrow{e_2} v_3 \cdots\cdots v_{n-1} \xrightarrow{e_{n-1}} v_n$$

Observe that A_n has one sink vertex only, namely v_n. The ideal of $L_K(A_n)$ generated by v_n is the whole algebra. The number of paths ending at v_n is n. Thus by lemma 2.0.2 we have $L_K(A_n) \cong M_n(K)$.

The sum of these ideals gives us the Lpa.

Lemma 2.0.3. *Let E be a finite acyclic graph. Let the sinks in E be $v_1, \ldots, v_a$. Given any field K, $L_K(E) \cong \sum_{i=1}^{a} I(v_i)$.*

Proof. Given any monomial $k\alpha\beta^* \in L_K(E)$, if $r(\alpha) = r(\beta) = v_i$ for some $1 \leq i \leq a$ then $k\alpha\beta^* \in I(v_i)$. Otherwise since $r(\alpha) = u$ is not a sink we can write $k\alpha\beta^* = k\alpha(\sum_{\substack{e \in E^1 \\ s(e)=u}} ee^*)\beta^*$, where the sum in this expression is not empty. This can be rewritten as $k\alpha\beta^* = k \sum_{\substack{e \in E^1 \\ s(e)=u}} \alpha e(\beta e)^*$. For each term in this sum if $r(\alpha e) = r(\beta e) = v_i$ for some $1 \leq i \leq a$ then $k(\alpha e)(\beta e)^* \in I(v_i)$. In this way we can continue until every term of the sum belongs to some $I(v_i)$. Hence we may conclude that $k\alpha\beta^* \in \sum_{i=1}^{a} I(v_i)$. From this we may infer the desired result. □

The product of these ideals is zero.

Lemma 2.0.4. *Let E be a finite acyclic graph with distinct sinks v and w. Then $I(v)I(w) = 0$.*

Proof. Take two non-zero monomials $p_1q_1^* \in I(v)$ and $p_2q_2^* \in I(w)$ with $r(p_1) = r(q_1) = v$ and $r(p_2) = r(q_2) = w$. Consider the product $x = (p_1q_1^*)(p_2q_2^*) = p_1yq_2^*$, where $y = q_1^*p_2$. But since $r(q_1) = v$ and $r(p_2) = w$, $y = 0$. Therefore $x = 0$. Observe that a general element in $I(v)$ can always be written as a linear combination of such monomials as $p_1q_1^*$. Similarly for $I(w)$ and $p_2q_2^*$. The desired result follows immediately. □

Putting the above lemmas together, we obtain a theorem that classifies all Lpas generated by a finite acyclic graph.

Theorem 2.0.3. *Let E be a finite acyclic graph with sinks $v_1, \ldots, v_a$. Let $n(v_i)$ be the number of paths ending at sink v_i. Then for any field K, $L_K(E) \cong \bigoplus_{i=1}^{a} M_{n(v_i)}(K)$*

Proof. Observe that $I(v_i)$ has a local identity given by $\sum_{r(p)=v_i} pp^*$. Combining lemma 2.0.3 and lemma 2.0.4 above with lemma A.0.1 and lemma A.0.2 from

Appendix A, gives that $L_K(E) \cong \bigoplus_{i=1}^{a} I(v_i)$ as rings. Combining this with lemma 2.0.2 gives us the desired result. □

Finally, is possible to generalise the notion of an acyclic graph as follows,

Definition 2.0.5. Let E be a directed graph with a cycle $c = c_1 \cdots c_n$ for some $n \geq 1$ and edges c_i. Suppose there is an edge $e \in E^1$ such that $s(e) = s(c_i)$ for some $1 \leq i \leq n$ and $r(e) \neq r(c_j)$ for any $1 \leq j \leq n$. Then we say that c has an *exit*. If there are no cycles with exit in E, we say that E has *no exits*.

In this chapter we established The Finite Dimension Theorem, theorem 2.0.3. It is possible to obtain similar results for graphs containing cycles without exit, although this is not required for our purposes.

Chapter 3

The Graded Ring Structure

In this chapter we examine the graded ring structure of Leavitt path algebras. This will allow us to prove theorems about the graph theoretic properties of certain classes of Lpas. It also gives us some useful tools for thinking about Lpas.

3.1 Graded Rings

The definition of a graded ring is as follows

Definition 3.1.1. Given some index set J, a *graded ring* is a ring that is a direct sum of Abelian groups R_i such that $R_i R_j \subseteq R_{i+j}$ for all $i, j \in J$. The index set J can be any monoid.

A graded algebra is an algebra that is graded as a ring. We also require the following

Definition 3.1.2. In a graded ring, elements of any factor R_i of the decomposition are called *homogeneous elements* of *degree* i. For any $x \in R_i$ we write $\deg(x) = i$.

Graded rings can have ideals that respect the grading as follows,

Definition 3.1.3. If I is an ideal of a graded ring R indexed by M then write $I_i = I \cap R_i$ for $i \in M$. We say that I is *homogeneous* if $I = \sum_{i \in M} I_i$.

We can obtain graded quotient rings as follows,

Lemma 3.1.1. *If R is a graded ring indexed by M and I is a homogeneous ideal in R, then R/I is a graded ring with gradation defined by $(R/I)_i = (R_i + I)/I$, $R/I \cong \bigoplus_{i \in M} (R/I)_i$*

Proof. Define $I_i = R_i \cap I$. Consider some $x \in R/I$ with $x = x_0 + I$ and $x_0 \in R$. Let $x_0 = \sum_{j \in J} r_j$ where $r_j \in R_j$ for some finite set $J \subseteq M$. Since I is homogeneous we can write

$$x = \sum_{j \in J} (r_j + I_j) + \sum_{i \in (M-J)} I_i$$

Note that the above expression for x_0 as a sum of r_i is unique. It follows that the above expression for x as a sum of (R_i/I_i) is also unique. It follows that $R/I \cong \bigoplus_{i \in M} (R_i/I_i)$.

By the second isomorphism theorem $I_i = R_i \cap I$ is an ideal of R_i and $(R_i + I)/I \cong R_i/(R_i \cap I)$ as groups for any $i \in M$. Therefore $(R/I)_i \cong R_i/I_i$ as groups. Thus $R/I \cong \bigoplus_{i \in M} (R/I)_i$.

Observe that for any $i, j \in M$, $r_i \in R_i$, and $r_j \in R_j$, $(r_i + I)(r_j + I) = r_i r_j + I$ and $r_i r_j \in R_{i+j}$. Thus

$$(R/I)_i (R/I)_j \subseteq (R/I)_{i+j}$$

The desired result follows immediately. □

Homogeneous ideals are those generated by homogeneous elements,

Lemma 3.1.2. *If I is an ideal of a graded ring R indexed by M and I is generated by finitely many homogeneous elements of R then I is a homogeneous ideal.*

Proof. Suppose I is generated by homogeneous elements $\{g_1, g_2, \cdots, g_n\}$ for some $n \geq 1$. Consider some element $x \in I$ and write $x = \sum_{i=1}^{n} a_i g_i b_i$ for $a_i, b_i \in R$, $1 \leq i \leq n$. Consider some $\alpha \in M$. Define

$$y_{i,\alpha} = \sum_{\alpha = \beta + \gamma + \deg(g_i)} (a_i)_\beta g_i (b_i)_\gamma$$

Then $x_\alpha = \sum_{i=1}^{n} y_{i,\alpha}$. Notice that since $g_i \in I$ it follows that $y_{i,\alpha} \in I$. Since $y_{i,\alpha}$ is homogeneous it follows that $y_{i,\alpha} \in I_\alpha$. Therefore $x_\alpha \in I_\alpha$. By definition $x = \sum_{\alpha \in M} x_\alpha$. The desired result follows. □

We assign a grading to a Leavitt path algebra as follows,

Definition 3.1.4. Let E be a graph and let $L_K(E)$ be the corresponding Lpa over a field K. Let p and q be paths in E with lengths $n(p)$ and $n(q)$ respectively. Consider the monomial $pq^* \in L_K(E)$. We say that this monomial has degree d, where $d = n(p) - n(q)$. If $x \in L_K(E)$ is a linear combination of monomials all of degree d then we say that x is a *homogeneous element* of *degree d*.

Lpas are graded rings. This proof is based on [5] definition 2.3.

Theorem 3.1.1. *Let E be a graph and let $L_K(E)$ be the corresponding Lpa over a field K. The $L_K(E)$ is a graded algebra with factors given as the sets of homogeneous elements as per definition 3.1.4.*

Proof. Setting $\deg(v) = 0$ for all $v \in E^0$, $\deg(e) = 1$ and $\deg(e^*) = -1$ for all $e \in E^1$ we obtain a natural $\mathbb{Z}$-grading on the free K-algebra generated by $\{v, e, e^* \mid v \in E^0, e \in E^1\}$. Since all the Lpa relations are homogeneous of degree 0, by lemma 3.1.2 the ideal generated by these relations is homogeneous. Thus by lemma 3.1.1 we have a natural $\mathbb{Z}$-grading on $L_K(E)$. This grading corresponds to the one given in definition 3.1.4. The desired result follows from this. □

3.2 Baer and Rickart conditions

The preceding results about graded rings can be used to prove some facts about Lpas having certain algebraic properties.

Definition 3.2.1. For a subset X of a ring A, the *right annihilator* $\mathrm{ann}_r(X)$ of X in A denotes the set of elements $a \in A$ such that $xa = 0$ for all $x \in X$. The *left annihilator* $\mathrm{ann}_l(X)$ is defined analogously. It is straightforward to check that $\mathrm{ann}_r(X)$ is a right and $\mathrm{ann}_l(X)$ a left ideal of A.

Annihilator sets are used in the definition of Rickart and Baer algebras.

Definition 3.2.2. A ring A is said to be *right Rickart* if $\mathrm{ann}_r(x)$ is generated by an idempotent for any $x \in A$. That is, for any $x \in A$, $\mathrm{ann}_r(x) = eA$ for some $e \in A$ such that $e^2 = e$. A ring is said to be *left Rickart* if the analogous condition holds for left annihilators of elements of A, and *Rickart* if it is both left and right Rickart.

The idempotent that generates the left and right annihilators of zero is a right and left identity respectively. Consequently Rickart rings must be unital.

Let A be a $*$-ring. A is left Rickart if and only if it is right Rickart since $\text{ann}_r(x)^* = \text{ann}_l(x^*)$ for any $x \in A$

Definition 3.2.3. A unital ring A is *Baer* if $\text{ann}_r(X)$ (equivalently $\text{ann}_l(X)$) is generated by an idempotent for any $X \subseteq A$.

While Rickart algebras come in left and right varieties, there is only a two sided Baer algebra. Credit goes to Roozbeh Hazrat who helped with this proof.

Theorem 3.2.1. *The Baer ring definition is left-right symmetric.*

Proof. Observe that $\text{ann}_r(\text{ann}_l(X)) = eA$ for some idempotent e. Thus $\text{ann}_l(X)eA = 0$. Since A is unital it follows that $\text{ann}_l(X)e = 0$. For any $x \in \text{ann}_l(X)$ we have $x = x(e+(1-e)) = xe+x(1-e)$. But $xe = 0$ therefore $x = x(1-e)$. Thus $\text{ann}_l(X) = A(1-e)$. Since $(1-e)$ is an idempotent $\text{ann}_l(X)$ is generated by an idempotent. □

We may prove a simple theorem about Lpas that are Rickart. This has been adapted from a proof in [7].

Theorem 3.2.2. *Let E be a directed graph. Then E^0 is finite if $L_K(E)$ is Rickart.*

Proof. Suppose $L_K(E)$ is Rickart and E^0 is infinite. Consider $u \in E^0$. By assumption $\text{ann}_r(u) = eL_K(E)$, where e is idempotent. Writing $e = a_1 + a_2 + \ldots + a_n$, with a_i monomials, since E^0 is infinite one can pick a vertex v different from u such that $ve = 0$. Since $v \neq u$, $uv = 0$, so $v \in \text{ann}_r(u)$. Thus $v = ex$ for some $x \in L_K(E)$. Multiplying this from the left by v we get $v = 0$ which is a contradiction. □

We show that Baer Lpas are precisely those where the graph is finite with no exits. This has been adapted from a proof in [7].

Theorem 3.2.3. *Let E be a directed graph. Then E is finite with no exits if and only if $L_K(E)$ is Baer.*

Proof.

Case 1: *Baer implies finite no exit* Let $A = L_K(E)$ be a Baer ring. Then $L_K(E)$ is also Rickart and by theorem 3.2.2 E^0 is finite.

Suppose E has a cycle $a = a_1a_2 \dots a_n$ that has an exit edge b so that $v = s(b) = s(a_1) = r(a_n)$. Consider the infinite set of non-zero, orthogonal idempotents $S = \{a^i bb^* a^{*i} \mid i = 0, 1, \dots\}$ and the subsets $S_1 = \{a^i bb^* a^{*i} \mid i \text{ is odd}\}$ and $S_2 = \{a^i bb^* a^{*i} \mid i \text{ is even}\}$. Since A is Baer, there is an idempotent e such that

$$\mathrm{ann}_r(S_1) = eA$$

Since $S_2 \subseteq \mathrm{ann}_r(S_1)$, we have $a^i bb^* a^{*i} = ex_i$, where i is any even number and $x_i \in A$. Multiplying this equation on the left by e, since e is an idempotent, we get

$$a^i bb^* a^{*i} = ea^i bb^* a^{*i}, i \text{ is even} \tag{3.1}$$

On the other hand since A is Rickart and therefore unital,

$$a^i bb^* a^{*i} e = 0, i \text{ is odd} \tag{3.2}$$

Since S consists of homogeneous elements of degree zero, writing e as a sum of homogeneous components, from equation (3.1), equation (3.2), and theorem 3.1.1 it follows that there is a homogeneous element of degree zero (call it e again) that satisfies equation (3.1) and equation (3.2). Note this e does not have to be idempotent. Write e as $k_1x_1y_1^* + k_2x_2y_2^* + \dots + k_nx_ny_n^*$, where x_j and y_j are paths of the same length with $r(x_j) = r(y_j)$, $s(x_j) = s(y_j) = v$, and $k_j \in K$ for $j = 1 \dots n$. Also write e as $e_1 + e_2 + e_3 + e_4$ where

e_1 is the sum of those $x_jy_j^*$ with $x_j = y_j = a^i$ for some $i = 0, 1, \dots$

e_2 is the sum of those $x_jy_j^*$ where only $y_j = a^i$ for some $i = 0, 1, \dots$

e_3 is the sum of those $x_jy_j^*$ where only $x_j = a^i$ for some $i = 0, 1, \dots$

e_4 is the sum of the rest of the monomials

Note that the relation $(e_3+e_4)a^i b = 0$ holds in A for sufficiently large i by the form of the terms in e_3 and e_4 and relation $(CK1)$. Similarly, $(e_2^* + e_4^*)a^i b = 0$ holds in A for sufficiently large i. That is, $(e_3 + e_4)a^i bb^* a^{*i} = 0$ and $(e_2^* + e_4^*)a^i bb^* a^{*i} = 0$ for sufficiently large i. Thus,

$(e_1 + e_2)a^i bb^* a^{*i} = a^i bb^* a^{*i}$ for sufficiently large even i, and

$(e_1^* + e_3^*)a^i bb^* a^{*i} = 0$ for sufficiently large odd i.

Choose an integer m larger than the length of x_j and y_j for all x_j appearing in e_2 and all y_j appearing in e_3. Then $a^m a^{*m} e_2 = a^m a^{*m} e_3^* = 0$. This implies that

$$a^m a^{*m} e_1 a^i b b^* a^{*i} = a^i b b^* a^{*i} \text{ for sufficiently large even } i > m,$$

$$a^m a^{*m} e_1^* a^i b b^* a^{*i} = 0 \text{ for sufficiently large odd } i.$$

Represent e_1 as $\sum_{l=1}^{t} k_l a^{\eta_l} a^{*\eta_l}$ for some positive integer t and natural numbers η_l, $l = 1, \ldots, t$. For large enough m and odd $i > m$,

$$\begin{aligned} 0 &= a^m a^{*m} e_1^* a^i b b^* a^{*i} \\ &= \sum_{l=1}^{t} k_l^* a^m a^{*m} a^i b b^* a^{*i} \\ &= (\sum_{l=1}^{t} k_l^*) a^i b b^* a^{*i} \end{aligned}$$

so $(\sum_{l=1}^{t} k_l^*) a^i b b^* a^{*i} a^i b = (\sum_{l=1}^{t} k_l^*) a^i b = 0$ holds in A. Since a path is a linearly independent element of A, $\sum_{l=1}^{t} k_l^* = 0$ and so $\sum_{l=1}^{t} k_l = 0$ also. For large enough m and even $i > m$,

$$a^i b b^* a^{*i} = a^m a^{*m} e_1 a^i b b^* a^{*i} = (\sum_{l=1}^{t} k_l) a^i b b^* a^{*i} = 0$$

Thus $0 = (a^i b b^* a^{*i}) a^i b = a^i b$ holds in A. So $(b^* a^{*i})(a^i b) = v = 0$ holds in A. This is a contradiction since vertices are linearly independent elements of A by theorem 2.0.2. Hence the assumption that there is a cycle with exit must be false.

Case 2: *Finite no exit implies Baer* The reader is directed to Lemma 14 of [7] for the proof of this case.

□

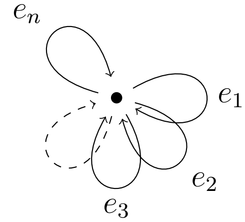
Chapter 4

The Invariant Basis Number

A free module is a module that has a basis. A free R-module M has the *invariant basis number property* if $M^n \cong M^m$ implies that $n = m$, for all positive integers n, m. This is true for certain classes of modules such as free modules over a commutative ring. It is not true in general, however. A ring or algebra has the IBN property if it has the IBN property as a module over itself. The question of whether a given Leavitt path algebra has the IBN property will be investigated here. We shall prove some theorems relating to this and show ways to construct infinite families of Lpas that have the IBN property and infinite families that lack the IBN property.

4.1 The n-petal graphs

To begin with we briefly look at an infinite family of non-IBN Leavitt path algebras. These are the n-petal graph Lpas.



Let E_n be the n-petal graph as depicted above. This is a graph with one vertex and n loop edges. Let the edges be $e_1, e_2, \ldots, e_n$. Let $R = L_K(E_n)$

be the Leavitt path algebra associated with E_n. Then there is a R-module isomorphism

$$\begin{aligned}\phi : R &\to R^n\\ x &\mapsto (xe_1, xe_2, \ldots, xe_n)\end{aligned}$$

Define

$$\begin{aligned}\psi : R^n &\to R\\ (x_1, x_2, \ldots, x_n) &\mapsto \sum_{k=1}^{n} x_k e_k^*\end{aligned}$$

It is not hard to see that both of the above maps are linear and n-linear respectively and therefore are homomorphisms. It is not difficult to check that the composition maps $\phi \circ \psi$ and $\psi \circ \phi$ give the identity on the generators of the corresponding algebras, thus they are the identity on the respective domains.

We therefore conclude that the n-petal graph Lpa is non-IBN for $n > 1$. Also this n is the smallest integer k such that $R \cong R^k$ although we do not give a proof here. The conclusion would be that R has module type $(1, n)$.

4.2 The graph monoid

Here is the definition of the graph monoid.

Definition 4.2.1. Let E be a graph with finitely many vertices. For any $v \in E^0$ consider the left $L_K(E)$-module $L_K(E)v$. Define $G(E)$ as the monoid generated by all the left modules $L_K(E)v$, where v ranges over all the vertices in the graph, with the monoid operation given by direct sum. Two elements of $G(E)$ are considered equal if they are isomorphic as left $L_K(E)$-modules.

Observe that for two distinct vertices $v, w \in E^0$ it is the case that

$$L_K(E)v \oplus L_K(E)w \cong L_K(E)(v + w)$$

as left $L_K(E)$-modules. The mapping is given by $(xv, yw) \mapsto (xv+yw)(v+w)$. The inverse map is $x(v+w) \mapsto (xv, xw)$. It is easy to see how this generalises to more than two distinct vertices. Furthermore observe that there is a module isomorphism

$$L_K(E)(\sum_{v\in E^0} v) \cong L_K(E)$$

since the sum of all vertices is the identity of the Lpa (theorem 2.0.1). It follows that

$$\bigoplus_{v\in E^0} L_K(E)v \cong L_K(E)$$

For brevity we may refer to $L_K(E)v \in G(E)$ as simply v.

Lemma 4.2.1. *Let E be a finite graph. Let $v \in E^0$ be a regular vertex. Let $s^{-1}(v) = \{e_1, \cdots, e_n\}$. Then $v = \sum_{i=1}^{n} r(e_i)$ in $G(E)$.*

Proof. We need to show that $L_K(E)v \cong \bigoplus_{i=1}^{n} L_K(E)r(e_i)$ as left $L_K(E)$-modules. The required map ϕ is given by

$$xv \mapsto (xve_ir(e_i))_{i=1}^n$$

Due to linearity of ϕ in x this map is a homomorphism. Define the map ψ on the co-domain as

$$(x_ir(e_i))_{i=1}^n \mapsto \sum_{i=1}^{n} x_ir(e_i)e_i^*$$

Due to the linearity in the x_i the map ψ is a homomorphism. It is not difficult to verify the composition of these maps in either order is the identity

$$\begin{aligned}(\psi\circ\phi)(xv) &\mapsto \sum_{i=1}^{n} xve_ir(e_i)e_i^* \\ &= \sum_{i=1}^{n} xe_ie_i^* \\ &= x\sum_{i=1}^{n} e_ie_i^* \\ &= xv\end{aligned}$$

and

$$
\begin{aligned}
(\phi \circ \psi)(x_i r(e_i))_{i=1}^n \mapsto & ((\sum_{j=1}^{n} x_j r(e_j) e_j^*) e_i r(e_i))_{i=1}^n \\
&= (x_i e_i^* e_i r(e_i))_{i=1}^n \\
&= (x_i r(e_i))_{i=1}^n
\end{aligned}
$$

It follows that ϕ is an isomorphism. □

The following definition plays an important role in understanding the IBN property of Lpas.

Definition 4.2.2. Let E be a finite graph. We denote by M_E the free abelian monoid on a set of generators $\{a_v \mid v \in E^0\}$, modulo the relations given by

$$a_v = \sum_{e \in E^1 | s(e)=v} a_{r(e)}$$

for each regular vertex v.

This theorem shows the role of M_E in the Lpa.

Theorem 4.2.1. *Let E be a finite graph. Then $G(E) \cong M_E$.*

Proof. Define the map

$$\phi : M_E \to G(E)$$

given on the generators by

$$a_v \mapsto L_K(E)v$$

It follows from lemma 4.2.1 that all the relations of M_E map to zero under ϕ. Hence this is a well-defined homomorphism. It is also clear that ϕ is surjective.

We also must show that ϕ is an injection. This is beyond the scope of this thesis. See [1] theorem 3.2.5. □

Observe that for any finite graph E and integer $k > 1$, the following relation in $G(E)$

$$\sum_{v \in E^0} v = k \sum_{v \in E^0} v$$

holds if and only if $R \cong R^k$ as R-modules, where $R = L_K(E)$. The conclusion is that for any graph E, we can determine if $L_K(E)$ is non-IBN by checking for the above property in $G(E)$.

Before we proceed we must introduce

Definition 4.2.3. Let M be any monoid. Let $(Z(M), +')$ be the free abelian group on the set M. Define the *group completion* of M, denoted $K(M)$, as the quotient of $Z(M)$ by the subgroup generated by $\{(x+'y)-'(x+y) \mid x, y \in M\}$. Here $+'$ and $-'$ denote the addition and subtraction in the free abelian group $Z(M)$.

We may now apply the graph monoid to determining whether a given Lpa has the IBN property.

Theorem 4.2.2. *Let E be a finite graph with vertices $v_1 \dots v_n$, where $n = \left|E^0\right|$. Define $\mathbf{c}$ to be the vector of size n containing only 1s. Define the vector*

$$\mathbf{v_i} = (b_{i1}, b_{i2}, \dots, b_{in}) - \mathbf{d_i}$$

where b_{ij} is the number of directed edges in E from v_i to v_j and $\mathbf{d_i}$ is the vector of size n that is zero everywhere except at position i which is 0 if v_i is a sink, and 1 otherwise. For any field K, $L_K(E)$ has the IBN property if $\mathbf{c} \notin \operatorname{span}\{\mathbf{v_i} \mid i = 1 \dots n\}$ over $\mathbb{Q}$.

Proof. Consider the $\mathbb{Z}$-module $\mathbb{Z}^n$ and the sub-module

$$A = \langle \mathbf{v_1}, \cdots, \mathbf{v_n} \rangle$$

Consider the quotient $\mathbb{Z}$-module

$$X = \mathbb{Z}^n / A$$

Let $K(M_E)$ be the group completion of the monoid M_E with monoid homomorphism

$$i : M_E \to K(M_E)$$

$$x \mapsto x$$

Any element of $Z(M_E)$ can be represented uniquely up to ordering of summation as

$$y_1 x_1 +' y_2 x_2 +' \cdots +' y_m x_m$$

for some $m \in \mathbb{N}$, where for $1 \leq i \leq m$, $y_i \in \mathbb{Z}$, $x_i \in M_E$, and for $1 \leq j \leq m$, $x_i = x_j \implies i = j$. Further, any x_i can be written as

$$x_i = x_{i,1}a_{v_1} + x_{i,2}a_{v_2} + \cdots + x_{i,n}a_{v_n}$$

where for $1 \leq j \leq n$, $x_{i,j} \in \mathbb{N}$. Note that this representation for x_i may not be unique in M_E. Given the above representations, there is a group homomorphism

$$j : K(M_E) \to X$$

$$\begin{pmatrix} y_1(x_{1,1}a_{v_1} + x_{1,2}a_{v_2} + \cdots + x_{1,n}a_{v_n})+' \\ y_2(x_{2,1}a_{v_1} + x_{2,2}a_{v_2} + \cdots + x_{2,n}a_{v_n})+' \\ \vdots \\ y_m(x_{m,1}a_{v_1} + x_{m,2}a_{v_2} + \cdots + x_{m,n}a_{v_n}) \end{pmatrix} \mapsto \begin{bmatrix} y_1x_{1,1} + y_2x_{2,1} + \cdots + y_mx_{m,1} \\ \vdots \\ y_1x_{1,n} + y_2x_{2,n} + \cdots + y_mx_{m,n} \end{bmatrix}$$

To see that j is well defined we must show that any element of the free group $Z(M_E)$ of the form

$$(x +' y) -' (x + y) \tag{4.1}$$

maps to zero under j. There are three cases to consider.

1. $x = y$. Write equation (4.1) as

$$\begin{pmatrix} 2(x)+' \\ -(2x) \end{pmatrix}$$

   It is easy to check that this maps to 0 under j.

2. $x \neq y$ and $x + y \neq x$ and $x + y \neq y$. Then write equation (4.1) as

$$\begin{pmatrix} (x_1a_{v_1} + x_2a_{v_2} + \cdots + x_na_{v_n})+' \\ (y_1a_{v_1} + y_2a_{v_2} + \cdots + y_na_{v_n})+' \\ -((x_1 + y_1)a_{v_1} + (x_2 + y_2)a_{v_2} + \cdots + (x_n + y_n)a_{v_n}) \end{pmatrix}$$

   Since all three summands x, y, and $x + y$ are mutually distinct, this expression is already in the required form. Thus we can apply j to get

$$\begin{bmatrix} x_1 + y_1 - (x_1 + y_1) \\ x_2 + y_2 - (x_2 + y_2) \\ \vdots \\ x_n + y_n - (x_n + y_n) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{0}$$

3. $x \neq y$ and $x + y = x$. Then equation (4.1) can be written as

$$\begin{aligned}(x +' y) -' (x + y) &= (x +' y) -' x \\ &= (x -' x) +' y \\ &= y\end{aligned}$$

Note that in the monoid M_E if $u \in M_E$ is an arbitrary element such that

$$u = u_1 a_{v_1} + u_2 a_{v_2} + \cdots + (u_i + 1) a_{v_i} + \cdots + u_n a_{v_n}$$

with all $u_j \in \mathbb{N}$ and $1 \leq i \leq n$ such that v_i is a regular vertex of E then there is a relation in the presentation of M_E

$$a_{v_i} = b_{i1} a_{v_1} + b_{i2} a_{v_2} + \cdots + b_{in} a_{v_n} \tag{4.2}$$

such that we can write

$$u = \sum_{k=1}^{n} (u_k + b_{ik}) a_{v_k}$$

This motivates the following definitions. Let e_i be the ith element of the standard basis for the semimodule $\mathbb{N}^n$. For each regular vertex v_i define R_i as the relation

$$(u_1, u_2, \cdots, u_n) + e_i \mapsto (u_1 + b_{i1}, u_2 + b_{i2}, \cdots, u_n + b_{in})$$

Let R be the union of all R_i such that v_i is a regular vertex. That is, $R = \bigcup_{v_i \in \mathrm{Reg}(E)} R_i$. If we take the symmetric, reflexive, transitive closure of R we get an equivalence relation $\overset{*}{\underset{R}{\longleftrightarrow}}$ on $\mathbb{N}^n$ such that $(\mathbb{N}^n / \overset{*}{\underset{R}{\longleftrightarrow}}) \cong M_E$. Write

$$\begin{aligned}x &= x_1 a_{v_1} + x_2 a_{v_2} + \cdots + x_n a_{v_n} \\ y &= y_1 a_{v_1} + y_2 a_{v_2} + \cdots + y_n a_{v_n}\end{aligned}$$

and define $x', y' \in \mathbb{N}^n$

$$\begin{aligned}x' &= (x_1, x_2, \cdots, x_n) \\ y' &= (y_1, y_2, \cdots, y_n)\end{aligned}$$

Now since $x + y = x$ in M_E this means that $x' + y' \overset{*}{\underset{R}{\longleftrightarrow}} x'$ in $\mathbb{N}^n$. Hence there must be a finite sequence of relations of the form R_i or R_i^{-1} for $1 \le i \le n$ whose composition relates x' to $x' + y'$ and visa-versa. Suppose such a sequence has length t. Let this sequence be

$$(s_1, s_2, \cdots, s_t)$$

That is,

$$x'(s_t \circ \cdots \circ s_2 \circ s_1)(x' + y') \tag{4.3}$$

For any integer z, denote by z^+ the positive part of z. That is,

$$z^+ = \max(0, z)$$

If $\mathbf{z}$ is a vector, then denote by $\mathbf{z}^+$ the vector obtained by taking the positive part of every coordinate of $\mathbf{z}$. Notice that R_i is a partial function and can be restricted to a total function as

$$r_i = R_i \restriction_{\mathbb{N}^n + (-\mathbf{v_i})^+}$$

so that

$$\begin{aligned} &r_i : (\mathbb{N}^n + (-\mathbf{v_i})^+) \to \mathbb{N}^n \\ &w \mapsto w + \mathbf{v_i} \end{aligned} \tag{4.4}$$

Similarly let

$$q_i = R_i^{-1} \restriction_{\mathbb{N}^n + \mathbf{v_i}^+}$$

so that

$$\begin{aligned} &q_i : (\mathbb{N}^n + \mathbf{v_i}^+) \to \mathbb{N}^n \\ &w \mapsto w - \mathbf{v_i} \end{aligned} \tag{4.5}$$

Notice that the composition $s_t \circ \cdots \circ s_2 \circ s_1$ can be restricted to a total function as

$$s = (s_t \circ \cdots \circ s_2 \circ s_1) \restriction_{\mathbb{N}^n + x'}$$

so that

$$\begin{aligned} &s : (\mathbb{N}^n + x') \to \mathbb{N}^n \\ &w \mapsto (s_t^0 \circ \cdots \circ s_2^0 \circ s_1^0)(w) \end{aligned}$$

where s_i^0 is the function obtained by restricting the domain of s_i as per equation (4.4) and equation (4.5). Observe that for each s_i^0 there is an

l such that $s_i^0 = r_l$ or $s_i^0 = q_l$. If $s_i^0 = r_l$ define $\rho_i = l$ and $\sigma_i = 1$. If $s_i^0 = q_l$ define $\rho_i = l$ and $\sigma_i = -1$. Define

$$\gamma = \sum_{i=1}^{t} \sigma_i \mathbf{v}_{\rho_\mathbf{i}}$$

We see that

$$s(x') = (s_t^0 \circ \cdots \circ s_2^0 \circ s_1^0)(x') = x' + \gamma$$

from equation (4.4) and equation (4.5), however,

$$s(x') = (s_t^0 \circ \cdots \circ s_2^0 \circ s_1^0)(x') = x' + y'$$

by equation (4.3). Therefore

$$x' + \gamma = x' + y'$$

Thus

$$y' = \gamma$$

since $\mathbb{N}^n$ is a cancellative monoid. Let $\gamma = (\gamma_1, \gamma_2, \cdots, \gamma_n)$. Thus

$$y = \gamma_1 a_{v_1} + \gamma_2 a_{v_2} + \cdots + \gamma_n a_{v_n}$$

Observe that $j(y) = \gamma$, however, $\gamma \in A$. It immediately follows that $\gamma = 0$ in X. Therefore $j(y) = 0$.

Due to symmetry the case $x \neq y$, $x + y = y$ is covered by case 3 above. There are no more cases to consider. Now we have established that j is a well defined homomorphism, define

$$\eta : M_E \to X$$
$$\eta = j \circ i$$

Suppose that the relation in $G(E)$

$$\sum_{v \in E^0} v = k \sum_{v \in E^0} v$$

for some fixed $k > 1$ holds. Then by theorem 4.2.1 the relation in M_E

$$\sum_{v \in E^0} a_v = k \sum_{v \in E^0} a_v$$

holds. Applying η to both sides we get

$$\mathbf{c} = k\mathbf{c}$$

in X. Therefore

$$(k-1)\mathbf{c} = 0$$

It follows that

$$(k-1)\mathbf{c} \in A$$

That is,

$$(k-1)\mathbf{c} = \lambda_1 \mathbf{v_1} + \cdots + \lambda_n \mathbf{v_n}$$

Hence

$$\mathbf{c} = \frac{\lambda_1}{k-1}\mathbf{v_1} + \cdots + \frac{\lambda_n}{k-1}\mathbf{v_n}$$

That is,

$$\mathbf{c} \in \text{span}\{\mathbf{v_i} \mid i = 1 \ldots n\} \text{ over } \mathbb{Q}$$

The contrapositive of this statement is that if $\mathbf{c} \notin \text{span}\{\mathbf{v_i} \mid i = 1 \ldots n\}$ then the relation $\sum_{v \in E^0} v = k \sum_{v \in E^0} v$ cannot hold. That is, R has the IBN property. □

Note that it is possible to check the condition for the IBN in theorem 4.2.2 efficiently using a computer. This amounts to computing the ranks of two integer matrices and comparing them. Computing ranks can be done efficiently using Gaussian Elimination.

It is possible to construct large classes of graphs whose Lpas have the IBN property. To do this we must first define the Cohn path algebra.

4.3 Cohn path algebras

These algebras are defined in section 1.5 of [1]. They are useful to us because they are closely related to Leavitt path alebras, however, Cohn path algebras always have the IBN property as well shall see.

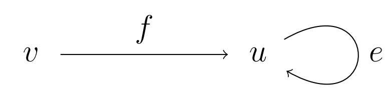
Definition 4.3.1. Let E be any directed graph and K any field. Define $(E^1)^* = \{e^* \mid e \in E^1\}$. The Cohn path algebra of E with coefficients in K, denoted $C_K(E)$, is the free associative K-algebra generated by the set $E^0 \cup E^1 \cup (E^1)^*$, subject to the relations $V, E1, E2, CK1$ given in definition 2.0.4.

Less formally a Cohn path algebra is determined by a directed graph and the first four out of the five Leavitt path algebra relations. The following construction and theorem makes explicit what the relationship is to Lpas. Both are adapted from [1].

Definition 4.3.2. Let E be a graph. We define E' as follows. Let Y be the regular vertices, i.e. the non-sink vertices of E. For each $v \in Y$ we consider a new symbol v' and for each $e \in r^{-1}(v)$ we consider a new symbol e'. Now define $E'^0 = E^0 \cup \{v' \mid v \in Y\}$ and $E'^1 = E^1 \cup \{e' \mid r(e) \in Y\}$. For each $e \in E^1$ define $r_{E'}(e) = r_E(e)$ and $s_{E'}(e) = s_E(e)$. Also define $s_{E'}(e') = s_E(e)$ and $r_{E'}(e') = r_E(e)'$.

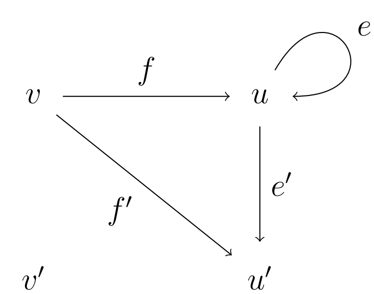
For example

E:



and

E':



The following theorem uses the above construction to provide us a useful K-algebra isomorphism.

Theorem 4.3.1. *Let E be any directed graph and K any field. Then*

$$C_K(E) \cong L_K(E')$$

Proof. We omit the details of this isomorphism and its proof since this is not germane to our immediate goals. A full account is available in Theorem 1.5.18 of [1]. □

We now have enough to prove that all Cohn path algebras are IBN.

Theorem 4.3.2. *Let E be any directed graph and K any field. Then $C_K(E)$ has the IBN property.*

Proof. We apply theorem 4.2.2 to $L_K(E')$. Let the vertices in E^0 be $v_1 \dots v_n$ where $n = |E^0|$. Let the vertices in E'^0 be $w_1 \dots w_{2n}$. We see that $w_i = v_i$ and $w_{i+n} = v_i'$ for $1 \leq i \leq n$. Let $\mathbf{v_i}$ for $1 \leq i \leq 2n$ be the vectors defined in theorem 4.2.2 applied to E'. Let $\mathbf{c}$ be the vector defined in theorem 4.2.2. We wish to show that $\mathbf{c} \notin \operatorname{span} \mathbf{v_i}$. Let $\mathbf{w_i}$ for $1 \leq i \leq n$ be the vectors from theorem 4.2.2 applied to E. From theorem 4.2.2 we have that $\mathbf{w_i} = (a_{i1}, a_{i2}, \dots, a_{in})^T - \mathbf{d_i}$ and due to the construction of E' we have

$$\mathbf{v_i} = \begin{bmatrix} \mathbf{w_i} \\ (a_{i1}, a_{i2}, \dots, a_{in})^T \end{bmatrix}$$

for $1 \leq i \leq n$. For $n < i \leq 2n$, $\mathbf{v_i} = \mathbf{0}$. We proceed with proof by contradiction.

Suppose that $\mathbf{c} \in \operatorname{span} \mathbf{v_i}$. Then $\mathbf{c} = \sum_{1 \leq i \leq n} \lambda_i \mathbf{v_i}$ for $\lambda_i \in \mathbb{Q}$. The jth row of this equation reads $1 = (\sum_{1 \leq i \leq n} \lambda_i a_{ij}) - \lambda_j$ for $1 \leq j \leq n$ and v_j non-sink. Due to the construction of E' we have $a_{ij} = a_{i(j+n)}$ for $1 \leq i, j \leq n$. So for $n < j \leq 2n$, $1 = \sum_{1 \leq i \leq n} \lambda_i a_{i(j-n)}$. Hence if $1 \leq j \leq n$ and v_j is a non-sink in E we have

$$\begin{aligned} 1 &= (\sum_{1 \leq i \leq n} \lambda_i a_{ij}) - \lambda_j \\ 1 &= \sum_{1 \leq i \leq n} \lambda_i a_{i(j+n)} = \sum_{1 \leq i \leq n} \lambda_i a_{ij} \\ \therefore 0 &= \lambda_j \end{aligned}$$

Thus we find that $\mathbf{c} = \sum_{i \in Y} \lambda_i \mathbf{v_i}$ where Y is the set of sinks in E^0. However if v_i is a sink then $\mathbf{v_i} = \mathbf{0}$ by definition. Therefore $\mathbf{c} = \mathbf{0}$ which is a contradiction. Therefore the assumption that $\mathbf{c} \in \operatorname{span} \mathbf{v_i}$ is false. So $\mathbf{c} \notin \operatorname{span} \mathbf{v_i}$. So by theorem 4.2.2 $L_K(E') \cong C_K(E)$ has the IBN property. □

This gives us a way to construct an infinite family of Leavitt path algebras having the IBN property, since every Cohn path algebra is isomorphism to some Lpa.

Chapter 5

Weighted Leavitt Path Algebras

These objects were first introduced by Roozbeh Hazrat in [5]. They provide more examples of non-IBN algebras. They also include the unweighted Lpas as a special case.

The ring given by $A = \{x_1, \ldots, x_n, y_1, \ldots y_n \mid XY = YX = 1\}$, where

$$X = \begin{bmatrix} x_1 & \ldots & x_n \end{bmatrix}, Y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

satisfies $A \cong A^n$ [10]. Leavitt path algebras embrace all of the above examples. The petal graph with n petals corresponds to the nth case. Consider now the ring given by $A = \{x_{11}, \ldots, x_{nm}, y_1, \ldots y_{mn} \mid XY = YX = 1\}$, where

$$X = \begin{bmatrix} x_{11} & \ldots & x_{1m} \\ \vdots & & \vdots \\ x_{n1} & \ldots & x_{nm} \end{bmatrix}, Y = \begin{bmatrix} y_{11} & \ldots & y_{1n} \\ \vdots & & \vdots \\ y_{m1} & \ldots & y_{mn} \end{bmatrix}$$

It can be shown that this satisfies $A^n \cong A^m$ [10]. Furthermore for $n, m > 1, n \neq m$, it can also be shown that these examples are not Leavitt path algebras. They are embraced, however, by weighted Leavitt path algebras.

In this section a number of proofs and examples involve output from a computer program written in the language Haskell. The code is available at the following URL https://github.com/rzil/wLpas

The following two definitions in this section are adapted from [5]

Definition 5.0.1. A *weighted graph* $E = (E^0, E^{st}, E^1, r, s, w)$ consists of three countable sets, E^0 called *vertices*, E^{st} called *structured edges*, and E^1 called *edges*, and maps $s, r : E^{st} \to E^0$, and a *weight map* $w : E^{st} \to \mathbb{N}$ such that $E^1 = \cup_{\alpha \in E^{st}} \{\alpha_i \mid 1 \le i \le w(\alpha)\}$, i.e., for any $\alpha \in E^{st}$, with $w(\alpha) = k$, there are k distinct elements $\{\alpha_1, \ldots, \alpha_k\}$, and E^1 is the disjoint union of all such sets for all $\alpha \in E^{st}$.

Definition 5.0.2. Let E be any weighted graph and K a field. The *weighted Leavitt path algebra* of E, denoted by $L_K(E, w)$ is the K-algebra generated by the sets E^0, $\{\alpha_1, \ldots, \alpha_{w(\alpha)} \mid \alpha \in E^{st}\}$, and $\{\alpha_1^*, \ldots, \alpha_{w(\alpha)}^* \mid \alpha \in E^{st}\}$ subject to the following relations.

1. $v_i v_j = \delta_{ij} v_i$ for every $v_i, v_j \in E^0$.
2. $s(\alpha)\alpha_i = \alpha_i r(\alpha) = \alpha_i$ and $r(\alpha)\alpha_i^* = \alpha_i^* s(\alpha) = \alpha_i^*$ for all $\alpha \in E^{st}$ and $1 \le i \le w(\alpha)$.
3. $\sum_{\{\alpha \in E^{st} \mid s(\alpha) = v\}} \alpha_i \alpha_j^* = \delta_{ij} v$ for fixed $1 \le i, j \le \max\{w(\alpha) \mid \alpha \in E^{st}, s(\alpha) = v\}$, for all $v \in E^0$.
4. $\sum_{1 \le i \le \max\{w(\alpha), w(\alpha')\}} \alpha_i^* \alpha_i' = \delta_{\alpha\alpha'} r(\alpha)$, for all $\alpha, \alpha' \in E^{st}$.

In the above, if $e \in E^{st}$ is a structured edge and $i > w(e)$ then $e_i = 0$.

Our first example shows how Lpas are a special case of wLpas.

Example 5.0.1. If (E, w) is a weighted graph such that $w(e) = 1$ for all $e \in E^{st}$, then $L_K(E, w)$ is isomorphic to the unweighted Leavitt path algebra $L_K(E)$.

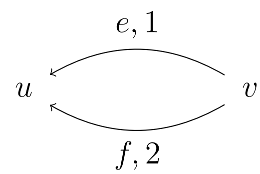
The next example is a weighted version of the n-petal graph Lpa.

Example 5.0.2. Let $n \ge 1$ and $k \ge 0$. If (E, w) is a weighted graph with precisely one vertex and precisely $n + k$ edges each of which has weight n, then $L_K(E, w)$ is isomorphic to the Leavitt algebra $L_K(n, n+k)$. For details see example 4 of [6].

For many weighted graphs, it is possible to construct unweighted graphs where the wLpa of the weighted graph is isomorphic to the Lpa of the unweighted graph; see [11] for some details about this.

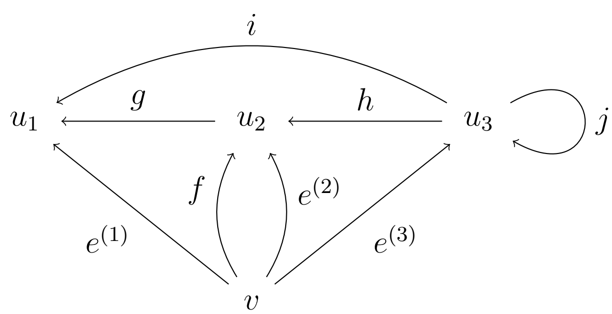
Example 5.0.3. This example comes from [12]. Consider the weighted graph

(E, w):



Consider also the unweighted graph

F:



There is a $*$-algebra isomorphism $\eta : L_K(E, w) \to L_K(F)$ mapping

$$\begin{aligned} v &\mapsto v & u &\mapsto \sum_{i=1}^{3} u_i \\ f_1 &\mapsto f & & \\ f_2 &\mapsto fg + e^{(1)}i^* + e^{(2)}h^* + e^{(3)}j^* & e_1 &\mapsto \sum_{i=1}^{3} e^{(i)} \end{aligned}$$

To verify the above example we wish to check that the map is well defined. Observe that any map from the generators of a free K-algebra to a ring induces a homomorphism at the level of K-algebra. It therefore suffices to check that the weighted relations hold under the map. This guarantees a map from the quotient of the free K-algebra modulo the weighted relations into the Lpa.

Here are a few examples of relations that must be checked. The following are examples of relations of type 3,

$$f_1 f_2^* = 0 \qquad f_2 f_1^* = 0$$

The following relations are of type 4,

$$e_1^* f_1 = 0 \qquad f_1^* e_1 = 0$$

The following relations are of type 3,

$$f_1 f_1^* + e_1 e_1^* = v \qquad f_2 f_2^* = v$$

The following relations are of type 4,

$$e_1^* e_1 = u \qquad f_1^* f_1 + f_2^* f_2 = u$$

The above relations are a subset of the relations for the wLpa, $L_K(E, w)$. We wish to check that the map η holds for all of the wLpa relations. The following is the output from a computer program doing this.

```
*Example1> testIsomorphism
(True,u.u + -1.u)
(True,u.v + 0)
(True,v.u + 0)
(True,v.v + -1.v)
(True,v.("e",1) + -1.("e",1))
(True,v.("f",1) + -1.("f",1))
(True,v.("f",2) + -1.("f",2))
(True,("e",1).u + -1.("e",1))
(True,("f",1).u + -1.("f",1))
(True,("f",2).u + -1.("f",2))
(True,u.("e",1)* + -1.("e",1)*)
(True,u.("f",1)* + -1.("f",1)*)
(True,u.("f",2)* + -1.("f",2)*)
(True,("e",1)*.v + -1.("e",1)*)
(True,("f",1)*.v + -1.("f",1)*)
(True,("f",2)*.v + -1.("f",2)*)
(True,("e",1).("e",1)* + ("f",1).("f",1)* + -1.v)
(True,("e",1).("e",2)* + ("f",1).("f",2)* + 0)
(True,("e",2).("e",1)* + ("f",2).("f",1)* + 0)
(True,("e",2).("e",2)* + ("f",2).("f",2)* + -1.v)
(True,("e",1)*.("e",1) + -1.u)
(True,("e",1)*.("f",1) + ("e",2)*.("f",2) + 0)
(True,("f",1)*.("e",1) + ("f",2)*.("e",2) + 0)
(True,("f",1)*.("f",1) + ("f",2)*.("f",2) + -1.u)
True
```

This program works by applying the map to each relation for the Lpa. Then the resulting expression is converted to basis form in the codomain. If the basis form is 0 it returns "True" for that particular relation. The last line says "True" which indicates that all the relations mapped to zero. Since all relations map to 0, we conclude that the mapping is a well defined homomorphism.

To verify that η is an isomorphism we must show it is surjective and injective. To show it is surjective it suffices to find, for each generator of $L_K(F)$, an element of $L_K(E,w)$ that maps to it. We start by observing that

$$\begin{aligned} v &\mapsto v \\ f_1 &\mapsto f \end{aligned}$$

Now,

$$\begin{aligned} f_1^* f_2 &\mapsto f^*(fg + e^{(1)}i^* + e^{(2)}h^* + e^{(3)}j^*) \\ &= f^* f g \\ &= g \end{aligned}$$

And,

$$gg^* = u_2$$

Therefore,

$$\begin{aligned} (f_1^* f_2)(f_1^* f_2)^* &= f_1^* f_2 f_2^* f_1 \\ &\mapsto gg^* \\ &= u_2 \end{aligned}$$

We have

$$(\sum e^{(i)})u_2 = e^{(2)}$$

Therefore,

$$\begin{aligned} e_1(f_1^* f_2 f_2^* f_1) &\mapsto (\sum e^{(i)})u_2 \\ &= e^{(2)} \end{aligned}$$

And

$$\begin{aligned} f_2^* e_1(f_1^* f_2 f_2^* f_1) &\mapsto (g^* f^* + ie^{(1)^*} + he^{(2)^*} + je^{(3)^*})e^{(2)} \\ &= he^{(2)^*}e^{(2)} \\ &= h \end{aligned}$$

And

$$\begin{aligned}(f_1^* f_2)^*(f_1^* f_2) &= f_2^* f_1 f_1^* f_2 \\ &\mapsto g^* g \\ &= u_1\end{aligned}$$

So

$$\begin{aligned}u - (f_2^* f_1 f_1^* f_2) - (f_1^* f_2 f_2^* f_1) &\mapsto \sum u_i - u_1 - u_2 \\ &= u_3\end{aligned}$$

Now

$$e_1 - e_1(f_1^* f_2 f_2^* f_1) \mapsto e^{(1)} + e^{(3)}$$

and

$$(e^{(1)^*} + e^{(3)^*})(fg + e^{(1)} i^* + e^{(2)} h^* + e^{(3)} j^*) = i^* + j^*$$

therefore

$$(e_1 - e_1(f_1^* f_2 f_2^* f_1))^* f_2 \mapsto i^* + j^*$$

however

$$u_3(i^* + j^*) = j^*$$

therefore let

$$\begin{aligned}A &= (e_1 - e_1(f_1^* f_2 f_2^* f_1))^* f_2 \\ B &= u - (f_2^* f_1 f_1^* f_2) - (f_1^* f_2 f_2^* f_1)\end{aligned}$$

So

$$\begin{aligned}(BA)^* &\mapsto (u_3(e^{(1)^*} + e^{(3)^*})(fg + e^{(1)} i^* + e^{(2)} h^* + e^{(3)} j^*))^* \\ &= (u_3(i^* + j^*))^* \\ &= j\end{aligned}$$

Now,

$$\begin{aligned}f_2(BA)^* &\mapsto (fg + e^{(1)} i^* + e^{(2)} h^* + e^{(3)} j^*) j \\ &= e^{(3)}\end{aligned}$$

Let

$$C = e_1(f_1^* f_2 f_2^* f_1)$$

then

$$\begin{aligned} e_1 - C - f_2(BA)^* &\mapsto (\sum e^{(i)}) - e^{(2)} - e^{(3)} \\ &= e^{(1)} \end{aligned}$$

Finally,

$$(e^{(1)^*}(fg + e^{(1)}i^* + e^{(2)}h^* + e^{(3)}j^*))^* = i$$

therefore,

$$\begin{aligned} ((e_1 - C - f_2(BA)^*)^* f_2)^* &= f_2^*(e_1 - C - f_2(BA)^*) \\ &\mapsto i \end{aligned}$$

We see that every generator of $L_K(F)$ occurs in the image of η. Therefore this must be a surjective map.

Define the map $\rho : L_K(F) \to L_K(E, w)$ on the generators by

$$\begin{aligned} v &\mapsto v & u_1 &\mapsto f_2^* f_1 f_1^* f_2 \\ u_2 &\mapsto f_1^* f_2 f_2^* f_1 & u_3 &\mapsto u - (f_2^* f_1 f_1^* f_2) - (f_1^* f_2 f_2^* f_1) \\ e^{(1)} &\mapsto e_1 - C - f_2(BA)^* & e^{(2)} &\mapsto C \\ e^{(3)} &\mapsto f_2(BA)^* & f &\mapsto f_1 \\ g &\mapsto f_1^* f_2 & h &\mapsto f_2^* C \\ i &\mapsto f_2^*(e_1 - C - f_2(BA)^*) & j &\mapsto (BA)^* \end{aligned}$$

It is a matter of computation to show that ρ is a well defined homomorphism. More computation can show that the composition maps $\rho \circ \eta$ and $\eta \circ \rho$ give the identity on the generators of the corresponding K-algebras, thus these maps are the identity on the respective domains. It follows that η is an isomorphism.

5.1 Nod paths

These objects are defined in [11] and they provide a method to compute a basis for a given wLpa. We require this in the following sections. First we must define the following

Definition 5.1.1. Let (E, w) be a weighted graph. For any regular vertex $v \in E^0$ set $w(v) = \max\{w(e) \mid e \in s^{-1}(v)\}$.

We also need the following concepts.

Definition 5.1.2. For any regular vertex $v \in E^0$ fix an $e^v \in s^{-1}(v)$ such that $w(e^v) = w(v)$. The e^v's are called *special edges.* The words in these sets

$$\{e^v_i(e^v_j)^* \mid v \in E^0, 1 \le i, j \le w(v)\}$$

and

$$\{e^*_1 f_1 \mid v \in E^0, e, f \in s^{-1}(v)\}$$

are called *forbidden.*

This leads to a definition of the nod-path.

Definition 5.1.3. A *nod-path* is any path of weighted edges and weighted ghost edges in (E, w) such that none of its subwords is forbidden.

Here we sketch a proof that the nod-paths span the wLpa.

Lemma 5.1.1. *Let (E, w) be a weighted graph. The set of nod-paths for this weighted graph span the corresponding wLpa $L_K(E, w)$.*

Heuristic Proof. Begin by observing that any expression involving multiplication, addition, and involution of vertices and edges with arbitrary brackets can always be expanded into a normal form involving a sum of paths with coefficients from an arbitrary field K. One way to convert to basis form is to find any forbidden subwords in a normal form of an expression and rewrite them. According to the wLpa relations

$$e^v_i(e^v_j)^* = \delta_{ij}v - \sum_{e \in s^{-1}(v), e \neq e^v} e_i e^*_j$$

and

$$e^*_1 f_1 = \delta_{ef} r(e) - \sum_{2 \le i \le \max\{w(e), w(f)\}} e^*_i f_i$$

After re-writing subwords in a normal form according to above rules, then expanding and reducing to a new normal form, the process is repeated. This continues until no forbidden subwords exist. A normal form containing no forbidden subwords consists of a sum of nod-paths. Assuming this process terminates, this implies that nod-paths span the wLpa. It also implies an algorithm to convert any expression into such a form. The only remaining piece of this proof would be to show that such an algorithm based on rewriting forbidden subwords must eventually terminate. It is not immediately obvious how to prove termination which is why this is only a sketch proof. Nevertheless this algorithm has been implemented in software, experimented with extensively, and it works in practise. □

The reason nod-paths are important is because they form a basis for the corresponding wLpa.

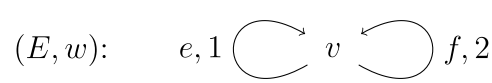
Theorem 5.1.1. *Let (E, w) be a weighted graph. The set of nod-paths for this weighted graph are linearly independent over K as elements of $L_K(E, w)$ and form a basis for the wLpa.*

Proof. We have sketched a proof that the nod-paths span the wLpa in lemma 5.1.1. It remains to show that they are linearly independent. A complete proof is given in [6], Theorem 16. □

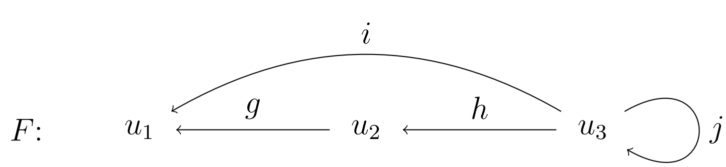
5.2 Some new homomorphisms

In this section I will give some new homomorphisms I have obtained.

Example 5.2.1. In [12], the author gives the following weighted graph,



At the time of writing this it is not known if the wLpa from this example is isomorphic to an unweighted Lpa. There is, however, an unweighted Lpa that admits a non-trivial homomorphism to this example.



The following map gives a $*$-algebra homomorphism

$$\eta : L_K(F) \to L_K(E, w)$$

$$\begin{aligned} u_1 &\mapsto f_2^* f_1 f_1^* f_2 \\ u_3 &\mapsto v - f_2^* f_1 f_1^* f_2 - f_1^* f_2 f_2^* f_1 \\ h &\mapsto f_2^* e_1 f_1^* f_2 f_2^* f_1 \\ i &\mapsto f_2^* e_1 f_2^* f_2 - f_2^* e_1 f_2^* e_1 e_1^* f_2 \end{aligned} \qquad \begin{aligned} u_2 &\mapsto f_1^* f_2 f_2^* f_1 \\ g &\mapsto f_1^* f_2 \\ j &\mapsto f_2^* e_1 f_2^* e_1 e_1^* f_2 \end{aligned}$$

This claim can be verified by checking that all Lpa relations hold using a computer. The following is the output from a computer program performing these checks.

```
*Example9> testHomomorphism
(True,u1.u1 + -1.u1)
(True,u1.u2 + 0)
(True,u1.u3 + 0)
(True,u2.u1 + 0)
(True,u2.u2 + -1.u2)
(True,u2.u3 + 0)
(True,u3.u1 + 0)
(True,u3.u2 + 0)
(True,u3.u3 + -1.u3)
(True,u2.("g",1) + -1.("g",1))
(True,u3.("h",1) + -1.("h",1))
(True,u3.("i",1) + -1.("i",1))
(True,u3.("j",1) + -1.("j",1))
(True,("g",1).u1 + -1.("g",1))
(True,("h",1).u2 + -1.("h",1))
(True,("i",1).u1 + -1.("i",1))
(True,("j",1).u3 + -1.("j",1))
(True,u1.("g",1)* + -1.("g",1)*)
(True,u2.("h",1)* + -1.("h",1)*)
(True,u1.("i",1)* + -1.("i",1)*)
(True,u3.("j",1)* + -1.("j",1)*)
(True,("g",1)*.u2 + -1.("g",1)*)
(True,("h",1)*.u3 + -1.("h",1)*)
(True,("i",1)*.u3 + -1.("i",1)*)
(True,("j",1)*.u3 + -1.("j",1)*)
```

```
(True,("g",1).("g",1)* + -1.u2)
(True,("h",1).("h",1)* + ("i",1).("i",1)* +
   ("j",1).("j",1)* + -1.u3)
(True,("g",1)*.("g",1) + -1.u1)
(True,("g",1)*.("h",1) + 0)
(True,("g",1)*.("i",1) + 0)
(True,("g",1)*.("j",1) + 0)
(True,("h",1)*.("g",1) + 0)
(True,("h",1)*.("h",1) + -1.u2)
(True,("h",1)*.("i",1) + 0)
(True,("h",1)*.("j",1) + 0)
(True,("i",1)*.("g",1) + 0)
(True,("i",1)*.("h",1) + 0)
(True,("i",1)*.("i",1) + -1.u1)
(True,("i",1)*.("j",1) + 0)
(True,("j",1)*.("g",1) + 0)
(True,("j",1)*.("h",1) + 0)
(True,("j",1)*.("i",1) + 0)
(True,("j",1)*.("j",1) + -1.u3)
True
```

The last line says "True" indicating that all relations were zero under the mapping. Thus the map is a well defined homomorphism.

We will find the image of η. Consider the following elements of $L_K(E,w)$

$$x = f_1^* f_2 \qquad y = f_2^* e_1 \qquad z = f_2^* f_2$$

Observe that x, y, z are all nod-paths of (E, w). Hence by theorem 5.1.1 they are linearly independent over K as elements of $L_K(E,w)$. The identity element of $L_K(E,w)$ can be expressed as $v = y^* y$. Therefore x, y, z generate a sub-algebra of $L_K(E,w)$. Let $B = \langle x, y, z \rangle$. It is easy to check that we can re-write the map η in terms of x, y, z

$$\begin{array}{lll} u_1 \mapsto x^* x & & \\ g \mapsto x & u_2 \mapsto x x^* & u_3 \mapsto y^* y - x^* x - x x^* \\ & h \mapsto y - yz & j \mapsto y^2 y^* \\ i \mapsto y(z - y y^*) & & \end{array}$$

This shows that $\operatorname{Im} \eta \subseteq B$. We may also express x, y, z in terms of g, h, i, j.

Observe the following

$$\begin{aligned} h + i + j &\mapsto (y - yz) + y(z - yy^*) + y^2 y^* \\ &= y(v - z) + y(z - yy^*) + y(yy^*) \\ &= y(v - z + z - yy^* + yy^*) \\ &= yv \\ &= y \end{aligned}$$

and

$$\begin{aligned} (h^* + i^* + j^*)(i + j) &\mapsto y^*(y(z - yy^*) + y^2 y^*) \\ &= y^* y(z - yy^*) + y^* yyy^* \\ &= z - yy^* + yy^* \\ &= z \end{aligned}$$

This shows that $B \subseteq \operatorname{Im} \eta$. Therefore $B = \operatorname{Im} \eta$. I claim without proof that η is an injection.

Lastly, we note that $L_K(F)$ is also a sub-algebra of the Lpa from example 5.0.3. There is a monomorphism to the Lpa from example 5.0.3 as an inclusion map.

The above motivates the following example,

Example 5.2.2. The Lpa for the following graph is known as the Toeplitz algebra

H:

Let F be the unweighted graph from example 5.2.1. There is a homomorphism $L_K(H) \to L_K(F)$ given on the generators as

$$\begin{aligned} u_1 &\mapsto u_1 + u_2 & a &\mapsto j \\ u_2 &\mapsto u_3 & b &\mapsto h + i \end{aligned}$$

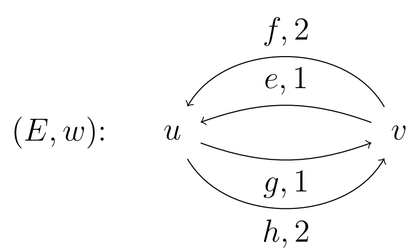
The validity of this homomorphism can be checked by verifying that all the Lpa relations hold under the map. This can be done either by human or computer.

```
*Example9> testToeplitzHomomorphism
(True,u1.u1 + -1.u1)
(True,u1.u2 + 0)
(True,u2.u1 + 0)
(True,u2.u2 + -1.u2)
(True,u2.("a",1) + -1.("a",1))
(True,u2.("b",1) + -1.("b",1))
(True,("a",1).u2 + -1.("a",1))
(True,("b",1).u1 + -1.("b",1))
(True,u2.("a",1)* + -1.("a",1)*)
(True,u1.("b",1)* + -1.("b",1)*)
(True,("a",1)*.u2 + -1.("a",1)*)
(True,("b",1)*.u2 + -1.("b",1)*)
(True,("a",1).("a",1)* + ("b",1).("b",1)* + -1.u2)
(True,("a",1)*.("a",1) + -1.u2)
(True,("a",1)*.("b",1) + 0)
(True,("b",1)*.("a",1) + 0)
(True,("b",1)*.("b",1) + -1.u1)
True
```

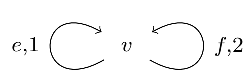
I claim without proof this map is injective.

To gain further insights we introduce another weighted graph,

Example 5.2.3. This is an original example.



We give two homomorphisms here as follows.

$\eta : L_K$ (

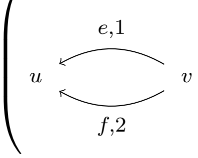
) $\to L_K(E, w)$

given on the generators as,

$$v \mapsto v + u \qquad f_1 \mapsto f_1 + h_1$$
$$e_1 \mapsto e_1 + g_1 \qquad f_2 \mapsto f_2 + h_2$$

and,

$$\rho : L_K(E, w) \to L_K \left(\quad \right)$$



given on the generators as,

$$u \mapsto u \qquad f_2 \mapsto f_2$$
$$v \mapsto v \qquad g_1 \mapsto f_2^*$$
$$e_1 \mapsto e_1 \qquad h_1 \mapsto f_1^*$$
$$f_1 \mapsto f_1 \qquad h_2 \mapsto e_1^*$$

One can verify that η is well defined by checking it holds for all relations of its domain

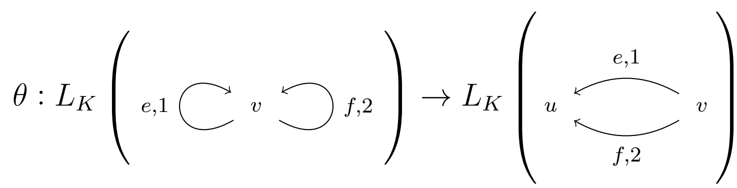
```
*Example11> testOriginalHomomorphism
(True,v.v + -1.v)
(True,v.("e",1) + -1.("e",1))
(True,v.("f",1) + -1.("f",1))
(True,v.("f",2) + -1.("f",2))
(True,("e",1).v + -1.("e",1))
(True,("f",1).v + -1.("f",1))
(True,("f",2).v + -1.("f",2))
(True,v.("e",1)* + -1.("e",1)*)
(True,v.("f",1)* + -1.("f",1)*)
(True,v.("f",2)* + -1.("f",2)*)
(True,("e",1)*.v + -1.("e",1)*)
(True,("f",1)*.v + -1.("f",1)*)
(True,("f",2)*.v + -1.("f",2)*)
(True,("e",1).("e",1)* + ("f",1).("f",1)* + -1.v)
(True,("e",1).("e",2)* + ("f",1).("f",2)* + 0)
(True,("e",2).("e",1)* + ("f",2).("f",1)* + 0)
(True,("e",2).("e",2)* + ("f",2).("f",2)* + -1.v)
```

```
(True,("e",1)*.("e",1) + -1.v)
(True,("e",1)*.("f",1) + ("e",2)*.("f",2) + 0)
(True,("f",1)*.("e",1) + ("f",2)*.("e",2) + 0)
(True,("f",1)*.("f",1) + ("f",2)*.("f",2) + -1.v)
True
```

I claim without proof that η is injective. We see that the generators for the codomain of ρ are in the range. Hence ρ is a surjective map. That is, ρ is an epimorphism. One can verify that ρ is well defined by checking it holds for all relations of $L_K(E, w)$.

```
*Example11> testOriginalEpimorphism
(True,v.v + -1.v)
(True,v.("e",1) + -1.("e",1))
(True,v.("f",1) + -1.("f",1))
(True,v.("f",2) + -1.("f",2))
(True,("e",1).v + -1.("e",1))
(True,("f",1).v + -1.("f",1))
(True,("f",2).v + -1.("f",2))
(True,v.("e",1)* + -1.("e",1)*)
(True,v.("f",1)* + -1.("f",1)*)
(True,v.("f",2)* + -1.("f",2)*)
(True,("e",1)*.v + -1.("e",1)*)
(True,("f",1)*.v + -1.("f",1)*)
(True,("f",2)*.v + -1.("f",2)*)
(True,("e",1).("e",1)* + ("f",1).("f",1)* + -1.v)
(True,("e",1).("e",2)* + ("f",1).("f",2)* + 0)
(True,("e",2).("e",1)* + ("f",2).("f",1)* + 0)
(True,("e",2).("e",2)* + ("f",2).("f",2)* + -1.v)
(True,("e",1)*.("e",1) + -1.v)
(True,("e",1)*.("f",1) + ("e",2)*.("f",2) + 0)
(True,("f",1)*.("e",1) + ("f",2)*.("e",2) + 0)
(True,("f",1)*.("f",1) + ("f",2)*.("f",2) + -1.v)
True
```

Denote the composition $\rho \circ \eta$ as θ. Then



is a homomorphism given on the generators as

$$\begin{aligned} v &\mapsto v + u & \qquad f_1 &\mapsto f_1 + f_1^* \\ e_1 &\mapsto e_1 + f_2^* & \qquad f_2 &\mapsto f_2 + e_1^* \end{aligned}$$

Since θ is the composition of two homomorphisms it must be itself a well defined homomorphism.

Now we will find a finite generating set for $\ker \rho$. Let $Q \subset L_K(E, w)$ be the set

$$Q = \{h_2^* - e_1, h_1^* - f_1, g_1^* - f_2\}$$

Let $I(Q)$ be the $*$-ideal generated by Q in $L_K(E, w)$. Let $S \subset L_K(E, w)$ be the set $S = \{u, v, e_1, f_1, f_2\}$. Then let $A = \langle S \rangle$. By the second isomorphism theorem $A + I(Q)$ is a sub-algebra of $L_K(E, w)$. We have

$$\{h_2^*, h_1^*, g_1^*\} \subset A + I(Q)$$

Hence

$$\{h_2^*, e_1, h_1^*, f_1, g_1^*, f_2, u, v\} \subset A + I(Q)$$

The set on the left hand side is the set of generators for $L_K(E, w)$. Therefore

$$A + I(Q) \cong L_K(E, w)$$

Now $\ker \rho$ is an ideal of $L_K(E, w)$ by the first isomorphism theorem. Since $Q \subset \ker \rho$ it follows that

$$I(Q) \subseteq \ker \rho$$

Now take any $k \in \ker \rho$. Since $k \in L_K(E, w)$, write $k = k_A + k_Q$ where $k_A \in A$ and $k_Q \in I(Q)$. We have

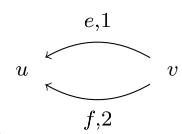
$$\begin{aligned} \rho(k) &= \rho(k_A + k_Q) \\ &= \rho(k_A) + \rho(k_Q) \\ &= \rho(k_A) \end{aligned}$$

However $\rho(k) = 0$. Therefore $\rho(k_A) = 0$. Once can show that the restriction

$$\rho_0 = \rho \restriction_A$$

is an isomorphism

$$A \xrightarrow{\sim} L_K \left(\quad \right)$$



It follows that $\rho_0(k_A) = 0$, and,

$$k_A = \rho_0^{-1}(0) = 0$$

Hence $k = k_Q$. That is, $k \in I(Q)$. We may conclude that $\ker \rho \subseteq I(Q)$. So

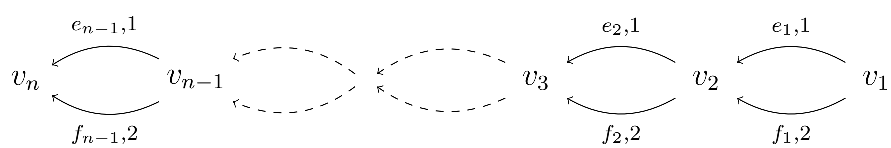
$$\ker \rho = I(Q)$$

Define the set

$$Q_0 = \{e_1 - f_2^*, f_1 - f_1^*\}$$

It is easy to see that $Q_0 \subset \ker \theta$. I claim without proof that $I(Q_0) = \ker \theta$.

Example 5.2.4. Consider the following weighted graph (G_n, w_{G_n}) for some fixed integer $n > 1$,

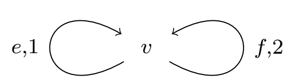


I claim there is a faithful representation,

$$\xi : L_K(G_n, w_{G_n}) \hookrightarrow M_R(n)$$

where,

$$R = L_K \left(\quad \right)$$



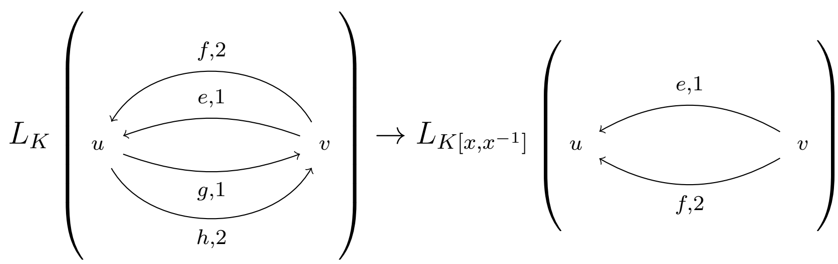
given by,

$$e_{i,1} \mapsto e_1 c_{i,i+1} \qquad f_{i,1} \mapsto f_1 c_{i,i+1} \qquad f_{i,2} \mapsto f_2 c_{i,i+1}$$
$$v_i \mapsto v c_{i,i}$$

where $c_{i,j}$ is the n by n single-entry matrix with 1 in the ith row and jth column.

Note that the codomain is a matrix $*$-ring, where $*$ is the matrix adjoint. The matrix adjoint is formed by applying $*$ to all elements of the transpose matrix.

Example 5.2.5. I claim there is a monomorphism,

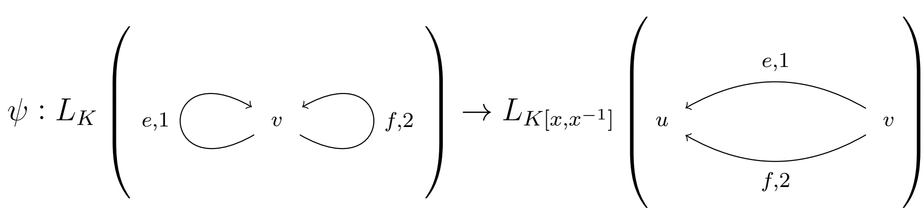


given by,

$$e_1 \mapsto xe_1 \qquad g_1 \mapsto xf_2^* \qquad u \mapsto u$$
$$f_1 \mapsto xf_1 \qquad h_1 \mapsto xf_1^* \qquad v \mapsto v$$
$$f_2 \mapsto xf_2 \qquad h_2 \mapsto xe_1^*$$

Note that in the codomain the coefficients come from a Laurent polynomial $*$-ring. We assume $x^* = x^{-1}$.

It is a consequence of example 5.2.3 and this example that we get a monomorphism,

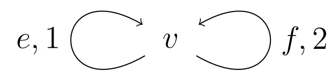


given as,

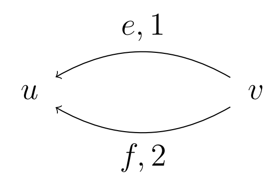
$$e_1 \mapsto x(e_1 + f_2^*) \qquad f_1 \mapsto x(f_1 + f_1^*) \qquad f_2 \mapsto x(f_2 + e_1^*)$$
$$v \mapsto u + v$$

Furthermore due to example 5.0.3 we see that the domain is sitting inside an unweighted Lpa over a Laurent polynomial $*$-ring.

Example 5.2.6. Define the weighted graph (E, w_E)



Define the weighted graph (G, w_G)



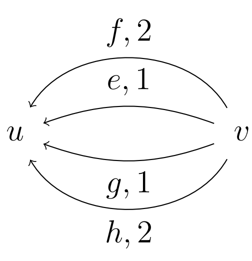
ξ from example 5.2.4 is a representation for (G_n, w_{G_n}) in matrices over $L_K(E, w_E)$. Thus ψ from example 5.2.5 allows us to obtain a representation for (G_n, w_{G_n}) in matrices over $L_{K[x,x^{-1}]}(G, w_G)$. One can show, however, that for $n > 0$ in this representation, the highest power of x that ever appears is x^{n-1}. Similarly, the lowest power is x^{-n+1}. Thus we can obtain a representation in matrices over $L_R(G, w_G)$, where $R = K[x]/\langle x^{2n-1} - 1\rangle$. One can easily show, however, that R itself has a faithful representation in $M_K(2n-1)$. The conclusion is that we get a representation as follows,

$$L_K(G_n, w_{G_n}) \to M_{L_K(G,w_G)}(2n^2 - n)$$

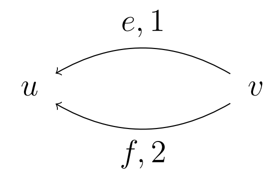
I claim that this is a faithful representation.

The next example is a generalisation of some of the above ideas,

Example 5.2.7. Consider the following weighted graph (H, w_H),



Let (G, w_G) be the weighted graph,



One has the following representation,

$$\phi : L_K(H, w_H) \to M_{L_K(G,w_G)}(3)$$

given as,

$$u \mapsto \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & u \end{bmatrix} \quad e_1 \mapsto \begin{bmatrix} 0 & 0 & e_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad g_1 \mapsto \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & e_1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$v \mapsto \begin{bmatrix} v & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad f_1 \mapsto \begin{bmatrix} 0 & 0 & f_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad h_1 \mapsto \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & f_1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$f_2 \mapsto \begin{bmatrix} 0 & 0 & f_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad h_2 \mapsto \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & f_2 \\ 0 & 0 & 0 \end{bmatrix}$$

Note, however, this representation is not faithful, since,

$$\phi(e_1)\phi(h_2^*) = 0$$

but

$$e_1 h_2^* \neq 0$$

It is not clear at this time whether it is possible to obtain a faithful representation for $L_K(H, w_H)$ in matrices over $L_K(G, w_G)$.

One can see that example 5.2.7 is an instance of the following conjecture,

Conjecture 5.2.1. *Consider any directed acyclic graph E. Replace every edge of E by the subgraph (G, w_G) from example 5.2.7. Call the resulting weighted graph (Q, w_Q). The wLpa $L_K(Q, w_Q)$ will have a representation in a direct sum of matrix rings over $L_K(G, w_G)$.*

We now look at some other properties of wLpas.

5.3 Gelfand-Kirillov dimension

The Gelfand-Kirillov dimension [13] is a wLpa invariant that is relatively easy to compute. Given some wLpa A and fixing a basis for this, define $d_V(n)$ to be the number of nod-paths of length at most n in A. Then

$$\operatorname{GKdim} A = \limsup_{n\to\infty} \log_n d_V(n) \tag{5.1}$$

This definition does not depend on the choice of basis of A. If $d_V(n) \sim n^k$ for some natural number k, then A is said to have polynomial growth and we have $\operatorname{GKdim} A = k$. If on the other hand $d_V(n) \sim a^n$ for some real number a, then A is said to have exponential growth and we have $\operatorname{GKdim} A = \infty$.

We will be able to compute this invariant in the next section.

5.4 Computations involving wLpas

We have already seen the output of some computations in earlier proofs. One may represent elements of a wLpa in a computer program and perform computations involving them. One may also reduce an expression to basis form, which gives a canonical representation for any element of an wLpa. Importantly this gives us a method for checking equality of expressions. In addition to this one may compute the basis for a wLpa. In case the basis is infinite, one may enumerate the basis. Enumeration of the basis by order of increasing path length gives us a way to estimate the Gelfand-Kirillov dimension (GK dimension) very quickly. Note that the standard definition of GK-dimension which is $\limsup_{n\to\infty} \log_n(d_V(n))$ typically converges very slowly. We can also generate all the wLpa relations for a graph. This is useful to check if a homomorphism defined on vertices and edges is well defined. All the above computations have been succinctly implemented in the programming language Haskell. The code is available at the URL given at the start of this section. The files from the source code repository given earlier, `Graph.hs` and `WeightedLpa.hs`, contain most of the relevant code for performing the above computations.

We can enumerate the first thirty basis elements for the wLpa from *example* 5.0.3 as follows.

```
*Example1> putStrLn $ unlines $ map show $ take 30 $\\
    WLpa.basis weighted_example
```

```
u
v
u.("e",1)*
v.("e",1)
u.("f",1)*
v.("f",1)
u.("f",2)*
v.("f",2)
u.("e",1)*.("f",2)
v.("e",1).("e",1)*
v.("e",1).("f",1)*
v.("e",1).("f",2)*
u.("f",1)*.("f",2)
v.("f",1).("e",1)*
u.("f",2)*.("e",1)
u.("f",2)*.("f",1)
u.("f",2)*.("f",2)
v.("f",2).("e",1)*
u.("e",1)*.("f",2).("e",1)*
v.("e",1).("e",1)*.("f",2)
v.("e",1).("f",1)*.("f",2)
v.("e",1).("f",2)*.("e",1)
v.("e",1).("f",2)*.("f",1)
v.("e",1).("f",2)*.("f",2)
u.("f",1)*.("f",2).("e",1)*
v.("f",1).("e",1)*.("f",2)
u.("f",2)*.("e",1).("e",1)*
u.("f",2)*.("e",1).("f",1)*
u.("f",2)*.("e",1).("f",2)*
u.("f",2)*.("f",1).("e",1)*
```

We now examine the GK-dimension using this software.

```
*Example1> map (WLpa.d_v weighted_example) [1..]
[8,18,32,50,72,98,128,162,200,^CInterrupted.
```

The above computation evaluates the d_V function for all positive integers in increasing order. We terminate the program after we have enough terms. We place these terms into a list as follows

```
*Example1> let ns = [8,18,32,50,72,98,128,162,200]
```

Now compute first differences

```
*Example1> let ns1 = zipWith (-) (tail ns) ns
*Example1> ns1
[10,14,18,22,26,30,34,38]
```

And second differences

```
*Example1> let ns2 = zipWith (-) (tail ns1) ns1
*Example1> ns2
[4,4,4,4,4,4,4]
```

From this we clearly see that the second differences have become constant and we may infer that the function d_V above is polynomial of degree two. Naturally this is not a rigorous proof but it provides strong numerical evidence that the GK dimension is 2.

The file from the source code repository give earlier, `Example9.hs`, contains the graph from Example 19 from [12]. It is not currently known if this example is isomorphic to an unweighted Lpa. We may examine the GK dimension as follows

```
*Example9> map (WLpa.d_v weighted_example) [1..]
[7,35,163,747,3411,15563,^CInterrupted.
```

If we compute fifth differences, we see the sequence has not stabilised yet

```
*Example9> let ns = map (WLpa.d_v weighted_example) [1..8]
*Example9> let f xs = zipWith (-) (tail xs) xs
*Example9> (iterate f ns) !! 5
[4516,20600,93968]
```

These numbers are increasing rapidly. This suggests that the GK dimension is at least five. In fact the GK dimension will be infinite in this case.

Idempotents and projections can give us useful information about *-rings. We may use this software to search for these. The search requires us to supply the field K. In this case we use the field $K = \mathbb{Z}_2$.

```
*Example1> let someProjections = take 10 $
   WLpa.projections weighted_example $ tail z2
```

We use the tail function to drop the 0 element from the field since this is not required in the search. Now print out the results

```
*Example1> putStrLn $ unlines $ map show someProjections
0
u
v
u + v
v.("e",1).("e",1)*
u + v.("e",1).("e",1)*
v + v.("e",1).("e",1)*
u + v + v.("e",1).("e",1)*
^CInterrupted.
```

This list will be infinite. In particular any element of the form $e^k e^{*k}$ will be a projection. We can perform a similar search for idempotents.

```
*Example1> let someIdempotents = take 50 $
   WLpa.idempotents weighted_example $ tail z2
```

Now print out the results

```
*Example1> putStrLn $ unlines $ map show someIdempotents
0
u
v
u + v
u + u.("e",1)*
v + u.("e",1)*
```

We omit some elements of the output for brevity. The following are more idempotents.

```
u + v.("e",1) + u.("f",1)* + v.("e",1).("f",1)*
v + v.("e",1) + u.("f",1)* + v.("e",1).("f",1)*
v.("e",1).("e",1)* + v.("e",1).("f",1)*
u + v.("e",1).("e",1)* + v.("e",1).("f",1)*
v + v.("e",1).("e",1)* + v.("e",1).("f",1)*
u + v + v.("e",1).("e",1)* + v.("e",1).("f",1)*
```

This list will be infinite. In particular any element of the form $e^k e^{*k}$ will be an idempotent.

In this chapter we have demonstrated the use of computer software to compute invariants of wLpas. We also have been able to check that certain homomorphisms are well defined. In section 5.2 we obtained some new homomorphisms and representations. We searched for idempotents and projections of an wLpa. Although not shown here one may also compute the module type using the graph monoid. Performing such computations by hand in many cases would be time consuming and difficult. It is the author's hope that software like this could be helpful in teaching this subject and in finding new results.

Further work might involve the following. A proof of termination for the algorithm in lemma 5.1.1 would establish that nod-paths form a spanning set for wLpas. The homomorphism η given in example 5.2.1 is claimed to be injective. Proving injectivity of η should be possible using some of the results obtained in example 5.2.3. Giving a rigorous proof for conjecture 5.2.1 might be useful.

Appendix A

Ring Theory

Here we prove some general ring theory results needed in other sections.

Lemma A.0.1. *Let R be a ring with identity. Suppose we have a finite set S of ideals I_k in R, indexed by positive integers k, such that $I_i I_j = 0$ for any $i \neq j$. Furthermore suppose that for each I_k there exists a "local identity" element $u_k \in I_k$ such that for any $x \in I_k$ we have that $u_k x = x u_k = x$. Then it is true that $\sum_{k=1}^{n} I_k \cong \bigoplus_{k=1}^{n} I_k$ as R-modules.*

Proof. Consider the map

$$f : \sum_{k=1}^{n} I_k \longrightarrow \bigoplus_{k=1}^{n} I_k$$

given by

$$\sum_{k=1}^{n} z_k \mapsto (z_k)_{k=1}^{n}$$

where $z_k \in I_k$ for all k. Clearly this is module a homomorphism, provided it is a well defined function. To show it is well defined we must show that for any $x, y \in \sum_{k=1}^{n} I_k$

$$x = y \implies f(x) = f(y)$$

Suppose that for some set of $z_k, w_k \in I_k$ we have

$$\sum_{k=1}^{n} z_k = \sum_{k=1}^{n} w_k$$

Then for any $1 \leq j \leq n$

$$u_j \sum_{k=1}^{n} z_k = u_j \sum_{k=1}^{n} w_k$$

And due to the mutual orthogonality of the ideals I_k it follows that

$$u_j z_j = u_j w_j$$

Hence because u_j is a local identity

$$z_j = w_j$$

Hence

$$(z_k)_{k=1}^{n} = (w_k)_{k=1}^{n}$$

That is,

$$f\left(\sum_{k=1}^{n} z_k\right) = f\left(\sum_{k=1}^{n} w_k\right)$$

So f is a well defined function. To show it is an isomorphism we must also show it is a bijection. It is clear f is surjective. To show that f is injective we must show that

$$f(x) = f(y) \implies x = y$$

This follows because

$$\begin{aligned} (z_k)_{k=1}^{n} = (w_k)_{k=1}^{n} &\implies \forall k : z_k - w_k = 0 \\ &\implies \sum_{k=1}^{n}(z_k - w_k) = 0 \\ &\implies \sum_{k=1}^{n} z_k = \sum_{k=1}^{n} w_k \end{aligned}$$

□

Lemma A.0.2. *As a corollary to lemma A.0.1, if in addition we have* $R = \sum_{I \in S} I$*, then we have a ring isomorphism* $R \cong \bigoplus_{I \in S} I$ *given by the same* f *as above, with component-wise multiplication in* $\bigoplus_{I \in S} I$*.*

Proof. To show that f is a ring isomorphism we only have to show that for any $x, y \in \sum_{I \in S} I$

$$f(xy) = f(x)f(y)$$

And this follows easily from the mutual orthogonality of the $I \in S$. $\square$

Appendix B

Background Materials

Included here are various definitions for mathematical terms that are used throughout this thesis. All of these terms are commonplace within mathematical literature, however, they are included here as a reference for any reader that requires it. We begin with a foundational issue, that of set theory.

Definition B.0.1. Informally, a *set* is a collection of distinct objects. When dealing with sets that are not finite in size, paradoxes may arise. For this reason, a more rigorous foundational theory may be required. Such foundational issues are beyond the scope of this thesis, however, the reader is referred to [9] for an in depth treatment of the subject.

Definition B.0.2. Given a pair of sets X, Y, there is a set called the *Cartesian product* $X \times Y = \{(x, y) \mid x \in X, y \in Y\}$, whose elements are called *ordered pairs*.

With an understanding of sets and Cartesian product we can introduce relations and functions. For our purposes the terms function and map are interchangeable.

Definition B.0.3. A *binary relation* R from X to Y is a subset of $X \times Y$.

Definition B.0.4. Let X and Y be sets. Intuitively, a *function* or *map* f is a process that associates each element of X to a single element of Y. Formally, a function or map is defined by a set $G \subseteq X \times Y$ such that for every $x \in X$ there is exactly one element $y \in Y$ such that $(x, y) \in G$. One may use the notation $f : X \to Y$ to say f is a function from X to Y or simply $X \to Y$

to refer to any function from X to Y. For any $x \in X$ and $y \in Y$ one may write $x \mapsto y$ to say x maps to y. Observe that a function is a special type of binary relation.

And some special types of functions.

Definition B.0.5. A function $X \to Y$ is *surjective* if for every element $y \in Y$, there is an element $x \in X$ such that $f(x) = y$.

Definition B.0.6. A function $X \to Y$ is *injective* or *one-to-one* if for any $x \in X$ and $y \in Y$, $f(x) = f(y)$ implies $x = y$.

Definition B.0.7. A function $X \to Y$ is *bijective* if it is both injective and surjective.

Definition B.0.8. A function $f : X \to X$ is an *involution* if $f(f(x)) = x$ for all $x \in X$.

Definition B.0.9. A *binary operation* on a set S is a map $S \times S \to S$.

We may now define some terms relating to structure of binary operations and their underlying sets.

Definition B.0.10. Let S be a set with binary operation $\cdot : S \times S \to S$. Then $i \in S$ is an *identity* if and only if for all $x \in S$, $i \cdot x = x \cdot i = x$.

One can show that identity elements are always unique as follows.

Theorem B.0.1. *Let S be a set with binary operation $\cdot$ and identity elements $i, j \in S$. By definition $i \cdot j = i$ but also $i \cdot j = j$. Therefore $i = j$.*

Due to the above theorem we may use the definite article "the" when referring to identity elements.

Definition B.0.11. Let S be a set with binary operation $\cdot : S \times S \to S$. An element $x \in S$ has an *inverse* x^{-1} if and only if $x \cdot x^{-1} = x^{-1} \cdot x = i$, where i is the identity.

Definition B.0.12. A binary operation $\cdot$ on a set S is *associative* if and only if for all $x, y, z \in S$, $x \cdot (y \cdot z) = (x \cdot y) \cdot z$.

Definition B.0.13. A binary operation $\cdot$ on a set S is *commutative* if and only if for all $x, y \in S$, $x \cdot y = y \cdot x$.

Definition B.0.14. Let S be a set and let $\cdot$ and $+$ be binary operations on S. Then $\cdot$ is *left distributive* over $+$ if and only if for all $x, y, z \in S$, $x \cdot (y + z) = x \cdot y + x \cdot z$.

Definition B.0.15. Let S be a set and let $\cdot$ and $+$ be binary operations on S. Then $\cdot$ is *right distributive* over $+$ if and only if for all $x, y, z \in S$, $(y + z) \cdot x = y \cdot x + z \cdot x$.

Definition B.0.16. Let S be a set and let $\cdot$ and $+$ be binary operations on S. Then $\cdot$ is *distributive* over $+$ if and only if it is both left distributive and right distributive.

This is sufficient to define some specific mathematical structures.

Definition B.0.17. A monoid is a set with a binary operation that satisfies associativity and has an identity element.

Definition B.0.18. A group is a monoid where every element has an inverse.

Definition B.0.19. An Abelian group is a group where the operation is commutative.

Definition B.0.20. A ring is a set R with two binary operations $+$ and $\cdot$ called *addition* and *multiplication*, respectively, satisfying the following

1. R is an Abelian group under addition with additive identity 0

2. R is a monoid under multiplication with multiplicative identity 1

3. Multiplication is distributive over addition

Definition B.0.21. A field is a ring where the multiplication is commutative and all elements except 0 have a multiplicative identity.

Definition B.0.22. Let G be a group and R a ring. A map $R \times G \to G$ is called *scalar multiplication* if and only if it obeys the following rules

1. Additivity in the scalar: $(c + d)\boldsymbol{v} = c\boldsymbol{v} + d\boldsymbol{v}$.

2. Additivity in the vector: $c(\boldsymbol{v} + \boldsymbol{w}) = c\boldsymbol{v} + c\boldsymbol{w}$.

3. Compatibility of product of scalars with scalar multiplication: $(cd)\boldsymbol{v} = c(d\boldsymbol{v})$.

4. Multiplication by 1 does not change a vector $1\boldsymbol{v} = \boldsymbol{v}$.

5. Multiplication by -1 gives the additive inverse $(-1)\boldsymbol{v} = \boldsymbol{-v}$.

Definition B.0.23. A module is an Abelian group V together with a ring R, and a scalar multiplication operation $R \times V \to V$.

Definition B.0.24. A vector space is module where the ring of scalars is a field.

Definition B.0.25. Let R be a ring. Then define the center of the ring R as $Z(R) = \{x \in R \mid \forall y \in R : xy = yx\}$.

Definition B.0.26. Let R be a ring. An element $x \in R$ is called *idempotent* if $x^2 = x$

Definition B.0.27. A ring A is said to be involutive, or a $*$-ring, if it has an involution $*$ such that $(xy)^* = y^*x^*$ for all $x, y \in A$

Definition B.0.28. Let R be a $*$-ring. An element $x \in R$ is called a *projection* if $xx^* = x$

The following definitions explain the notion of basis of a module.

Definition B.0.29. Let R be a ring and let M be an R-module. A finite subset $S = \{v_1, v_2, \cdots, v_n\}$ of M is said to be *linearly dependent* if there exist scalars $a_1, a_2, \cdots, a_n \in R$ not all zero, such that

$$a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0$$

The set S is said to be *linearly independent* otherwise.

Definition B.0.30. Let R be a ring and let M be an R-module. A finite subset $S = \{v_1, v_2, \cdots, v_n\}$ of M is said to *span* M if for any $x \in M$ there exist scalars $a_1, a_2, \cdots, a_n \in R$ such that $x = a_1v_1 + a_2v_2 + \cdots + a_nv_n$. S is also said to *generate* M.

Definition B.0.31. Let R be a ring and let M be an R-module. A finite subset $S \subseteq M$ is said to be a *basis* for M if it is linearly independent over M and it also spans M.

Now we give definitions for some special types of maps between structures.

Definition B.0.32. A *homomorphism* is a map between two structures of the same type that preserves the operations of the two structures. For example between two groups, rings, modules, or algebras. This means a map $f : A \to B$ between two sets A, B equipped with the same structure such that, if $*_A$ is a binary operation on A then there is a binary operation $*_B$ on B such that for all $x, y \in A$, $f(x *_A y) = f(x) *_B f(y)$. In addition to this the map f must preserve identity elements. So if i_A is the identity with respect to $*_A$ in A then $f(i_A)$ is the identity with respect to $*_B$ in B.

This allows us to define a new structure.

Definition B.0.33. An algebra is a unital ring R together with a field F and a homomorphism $\eta : F \to Z(R)$

Definition B.0.34. Let R and S be rings and let f be a ring homomorphism from R to S. The *kernel* of f is denoted $\ker(f)$. It is defined as the set

$$\ker(f) = \{r \in R : f(r) = 0_S\}$$

Definition B.0.35. A *monomorphism* is an injective homomorphism.

Definition B.0.36. An *isomorphism* is a bijective homomorphism.

We require the notion of direct sum. Informally, the direct sum of two structures (for example modules) is the Cartesian product of the underlying sets combined with the most "natural" lifting of the operations on those structures to the Cartesian product.

Definition B.0.37. Let R be a ring and $\{M_i : i \in I\}$ a set of left R-modules indexed by a set I. The *direct sum* of these modules, written $\bigoplus_{i \in I} M_i$, is defined to be the set of all sequences (α_i) where $\alpha_i \in M_i$ and $\alpha_i = 0$ for all but finitely many indices i. The set inherits the module structure via component wise addition and scalar multiplication.

The notion of a free algebra is needed.

Definition B.0.38. For a commutative ring R, the *free algebra* on n indeterminates $X = \{x_1, \cdots, x_n\}$ is the R-module with a basis consisting of all words over the alphabet X, including the empty word. This R-module becomes an R-algebra by defining a multiplication as follows: the product of any two basis elements is given by concatenation.

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